Varieties of distributive ℓ-pregroups

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Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(G, \wedge, \vee)$ is a lattice,
- $(G, \cdot, ^{-1}, 1)$ is a group $(x^{-1}x = 1 = xx^{-1})$ and
- multiplication preserves the order. (eqv: it distributes over join/over meet.)

A *lattice-ordered pregroup*, or *ℓ-pregroup*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, ^{\ell}, ^{r}, 1)$ where

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and $x^{\ell}x \leqslant 1 \leqslant xx^{\ell}$ and $xx^{r} \leqslant 1 \leqslant x^{r}x$, and
- multiplication preserves the order.

ℓ-groups $=$ ℓ-pregroups that satisfy $x^{\ell} = x^{r}$. In general, *n-periodicity*: $x^{\ell^n} = x^{r^n}$.

If we do not require a lattice order, then we obtain *pregroups*, which are used in mathematical linguistics, both theoretical and applied (Lambek, Buzskowski).

ℓ-pregroups are exactly *involutive residuated lattices* that satisfy $(xy)^{\ell} = y^{\ell}x^{\ell}$ (equivalently multiplication equals its DeMorgan dual: $xy = (y^{\ell}x^{\ell})^{r}$).
Residuated lattices include structures such as Boolean algebras, relation algebras, ideal lattices of rings, Heyting algebras and MV-algebras. Also, they form algebraic semantics for substructural logics, including many-valued, relevance, and linear logic.

LG : the variety of ℓ-groups; LP : ℓ-pregroups; $\mathsf{LP_n}$: n-periodic; DLP : distributive.
We have, $\mathsf{LP}_n \subseteq \mathsf{LP}_{kn}$. $\mathsf{LP}_1 = \mathsf{LG} \subseteq \mathsf{DLP} \subseteq \mathsf{LP}$. **[G.-Jipsen, AU 2012]** $\mathsf{LP_n} \subseteq \mathsf{DLP}$.

Example of distributive ℓ-pregroups and representation

[Cayley, PM 1854] Every group can be embedded in $\mathbf{S}_C$, for some set C.

[Holland, MMJ 1963] Every ℓ-group can be embedded in $\mathbf{Aut}(\mathbf{C})$, for some chain $\mathbf{C}$.

Definition: Given functions $f : \mathbf{P} \to \mathbf{Q}$ and $g : \mathbf{Q} \to \mathbf{P}$ between posets, we say that g is a *residual* for f, or that f is a *dual residual* for g, or that $\underline{(f, g) \text{ form a } \textit{residuated pair}}$, if

$$f(p) \leqslant q \ \Leftrightarrow \ p \leqslant g(q), \text{ for all } p \in P, q \in Q.$$

Fact: The residual of f, when it exists, is unique and we denote it by f^r. The dual residual of f, when it exists, is unique and we denote it by f^ℓ. Also,

$$f^\ell(y) = \bigwedge\{x : y \leqslant f(x)\} \qquad \text{and} \qquad f^r(y) = \bigvee\{x : f(x) \leqslant y\}.$$

Higher-order (dual) residuals: If it exists, f^{rrr} is called the residual of 3rd order, etc.

[GJKO, 2007] For every chain $\mathbf{C}$, the functions on $\mathbf{C}$ with all residuals (and dual) form a distributive ℓ-pregroup $\mathbf{F}(\mathbf{C})$, under composition, id_C, inverses, and pointwise order.

[G.-Jipsen, AU 2012] Every *distributive* ℓ-pregroup can be embedded in $\mathbf{F}(\mathbf{C})$, for some chain $\mathbf{C}$. **[G.-Gallardo, JoA 2024]** $\mathbf{C}$ can be taken to $\mathbf{J} \overrightarrow{\times} \mathbb{Z}$, for some chain $\mathbf{J}$.

[Holland, PAMS 1976] $\mathsf{LG} = \mathsf{V}(\mathbf{Aut}(\mathbb{Q}))$. **[G.-Gallardo, JoA 2024]** $\mathsf{DLP} = \mathsf{V}(\mathbf{F}(\mathbb{Z}))$.

Periodic ℓ-pregroups

Fact. The functions f of $\mathbf{F}(\mathbb{Z})$ that satisfy $f^{\ell^n} = f^{r^n}$ are exactly the n-periodic ones: $f(x+n) = f(x) + n$ for all $x \in \mathbb{Z}$.

Fact. The n-periodic elements $x^{\ell^n} = x^{r^n}$ of an ℓ-pregroup form a subalgebra.

We denote by $\mathbf{F}_n(\mathbf{C})$ the n-periodic subalgebra of $\mathbf{F}(\mathbf{C})$. (So, $\mathbf{F}_n(\mathbf{C}) \in \mathsf{LP}_n$.)

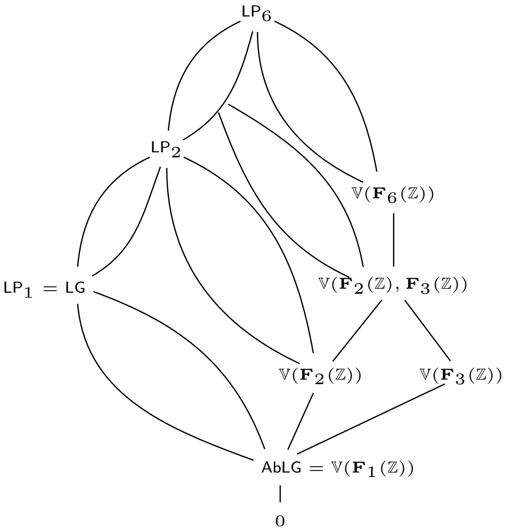
[G.-Gallardo, JoA 2025]

- G : S_C, LG : $\mathbf{Aut}(\mathbf{C})$, DLP : $\mathbf{F}(\mathbf{C})$, $\mathsf{LP_n}$: $\mathbf{F}_\mathsf{n}(\mathbf{C})$. $\mathbf{F}_1(\mathbf{C}) = \mathbf{Aut}(\mathbf{C})$
  Every algebra in $\mathsf{LP_n}$ can be embedded in $\mathbf{F}_n(\mathbf{J}\overrightarrow{\times}\mathbb{Z})$ for some chain $\mathbf{J}$.
  Moreover, $\mathbf{F}_n(\mathbf{J}\overrightarrow{\times}\mathbb{Z}) \cong \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$. (wreath product)
- $\bigvee_n \mathsf{LP_n} = \mathsf{DLP} = \bigvee_n \mathsf{V}(\mathbf{F}_n(\mathbb{Z})) = \mathsf{V}(\mathbf{F}_{\mathsf{fs}}(\mathbb{Z}))$. ([G.-G. 2024] DLP $= \mathsf{V}(\mathbf{F}(\mathbb{Z}))$)
  Note: $\mathsf{V}(\mathbf{F}_n(\mathbb{Z})) \neq \mathsf{LP_n}$ for all n. (ALG $= \mathsf{V}(\mathbf{Aut}(\mathbb{Z})) \neq \mathsf{V}(\mathbf{Aut}(\mathbb{Q})) = \mathsf{LP}_1 =$ LG.)
- $\mathsf{LP_n} = \mathsf{V}(\mathbf{F}_n(\mathbb{Q}\overrightarrow{\times}\mathbb{Z}))$.

The equational theory is decidable for:

- ℓ-groups **[Holland-McCleary HJM 1979]**
- DLP **[G.-Gallardo, JoA 2024]**
- all $\mathsf{LP_n}$ and $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ **[G.-Gallardo, JoA 2025]**.

Part of the subvariety lattice of LP_6



Theorem [G.-Santschi JoA 2026] The variety $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ is axiomatized relative to LP_n by $xx^{-1} = 1 \;\Rightarrow\; x^n y = yx^n$ ("n-th powers of group elements are central") or, equivalently, by $\sigma_n(x)^n y \approx y\sigma_n(x)^n$, where $\sigma_n(x) \coloneqq x \wedge x^{\ell\ell} \wedge \ldots \wedge x^{\ell^{2n-2}}$.

Theorem [G.-Santschi JoA 2026]

- The FSIs of $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ are exactly the n-periodic ℓ-pregroups with a totally ordered abelian group skeleton.
- The finitely generated FSIs of $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ are of the form $\mathbf{H}\overrightarrow{\times}\mathbf{F}_k(\mathbb{Z})$ with $k \mid n$ and $\mathbf{H}$ a finitely generated totally ordered abelian ℓ-group.
- Explicit *finite* axiomatization for any join of varieties of the form $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$.
- The set of proper subvarieties of the form $\bigvee_{q\in S} \mathsf{V}(\mathbf{F}_q(\mathbb{Z}))$, where $S \subseteq \mathbb{Z}^+$ constitutes an ideal in the subvariety lattice of DLP isomorphic to the lattice of finite downsets of the divisibility lattice of $\mathbb{Z}^+$.
- The periodic subvarieties of DLP organize into disjoint intervals of *properly* n-periodic varieties: If n is a prime power, this interval is $[\mathsf{V}(\mathbf{F}_n(\mathbb{Z})), \mathsf{LP}_n]$. In general, the bottom is $\bigvee_{q\in Q} \mathsf{V}(\mathbf{F}_q(\mathbb{Z}))$, where $Q = \{p_1^{k_1}, \ldots, p_l^{k_l}\}$ and $n = p_1^{k_1} \cdots p_l^{k_l}$ is the prime decomposition of n.

Non-periodic varieties

Question. How about the non-periodic subvarieties of DLP?

Theorem. [G.-Santschi] All *proper* subvarieties of DLP are periodic!

Corollary. The proper subvarieties of DLP organize into disjoint intervals: $[\bigvee_{q\in Q} \mathsf{V}(\mathbf{F}_q(\mathbb{Z})), \mathsf{LP}_n]$, where Q is the set of maximal prime powers of n.

In other words we prove:
Theorem. If $\mathbf{L}$ is a non-periodic distributive ℓ-pregroup, then $\mathsf{V}(\mathbf{L}) = \mathsf{DLP}$.

Proof sketch. Then, $\mathbf{L}$ contains periodic elements of arbitrary big periodicity or a non-periodic element. **Claim.** These elements can be taken to positive and idempotent.

Proof sketch of claim. In case $f \in \mathbf{L} \leqslant \mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$, for some chain $\mathbf{J}$, is non-periodic, and further $\widetilde{f} : J \to J$ is a bijective function, then $f = (\widetilde{f}, \overline{f}) \in \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}(\mathbb{Z})$ and $\forall m \exists i$: $\overline{f}_i$ is not m-periodic. $\forall k \in \mathbb{Z}$, $f^{[k]} := f^{(\ell\ell)^k}$, the global component of $g := f^{[k]} f^\ell$ is id:
$f^{[k]} f^\ell = (\widetilde{f}, \overline{f})^{[k]} (\widetilde{f}, \overline{f})^\ell = (\widetilde{f}, \overline{f}^{[k]})(\widetilde{f}^{-1}, \overline{f}^\ell \otimes \widetilde{f}^{-1}) = (id, (\overline{f}^{[k]} \otimes \widetilde{f}^{-1}) \cdot (\overline{f}^\ell \otimes \widetilde{f}^{-1}))$.
So, g is determined by $\overline{g} = (\overline{g}_i)_{i \in J} \in F(\mathbb{Z})^J$. We show that $\exists k$: $\overline{g}_i$ is not m-periodic.

Then we argue that there exists a non m-periodic positive idempotent $a \in \langle \overline{g}_i \rangle$, so there is a term t such that $a = t(\overline{g}_i)$. For $h := t(g)$, $hh^\ell \in \langle g \rangle$ is a positive idempotent with $\overline{hh^\ell}_i = aa^\ell = a$. Since a is not m-periodic, also hh^ℓ is not m-periodic.

Simulation

For $f, g \in \mathrm{F}(\mathbb{Z})$ and $k \in \mathbb{N}$, we write $f \sim_k g$ if f and g agree on $[-k, k]$.

We say that $\mathbf{L} \leqslant \mathbf{F}(\mathbb{Z})$ *simulates* $f \in \mathrm{F}(\mathbb{Z})$ if $\forall k \in \mathbb{N}, \exists g \in L : f \sim_k g$.

We say that $\mathbf{L} \leqslant \mathbf{F}(\mathbb{Z})$ *simulates* $\mathbf{M} \leqslant \mathbf{F}(\mathbb{Z})$ if $\mathbf{L}$ simulates each $f \in M$.

$\mathbf{L} \leqslant \mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ *simulates* $\mathbf{M} \leqslant \mathbf{F}(\mathbb{Z})$ if $\exists j \in J, \forall h \in M, \forall k \in \mathbb{N}, \exists g \in L : \overline{g}_j \sim_k f \ \& \ \widetilde{g} = id$.

Theorem. If $\mathbf{L} \leqslant \mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ simulates $\mathbf{M} \leqslant \mathbf{F}(\mathbb{Z})$, then $\mathsf{V}(\mathbf{M}) \subseteq \mathsf{V}(\mathbf{L})$.

Theorem. If $\mathbf{L} \leqslant \mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ simulates a finite-support function, then $\mathsf{V}(\mathbf{L}) = \mathsf{DLP}$.

Theorem. If $\mathbf{L} \leqslant \mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ simulates a fnction of periodicity n, $\mathsf{V}(\mathbf{F}_n(\mathbb{Z})) \subseteq \mathsf{V}(\mathbf{L})$.

Thank you!