Almost Everywhere is all you need

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Outline

Something happens **almost everywhere** if it happens on a closed unbounded set. Can also formulate using (partial) elementary submodels of the universe

1. Warmup: using elem. submodels to reprove Kaplansky's Theorem
2. **Cellular (cofibrant) generation** of a class of morphisms (Quillen)
    - A new characterization of cellular generation in terms of elementary submodels of the universe
    - Applications (old and new)

Outline

1. Part 1: set-theoretic elementary submodels warmup: Kaplansky's Theorem

2. Part 2: Cellular generation, a characterization, and applications

(partially) elementary submodels of universe

$(V, \in)$: universe of all sets (a model of ZFC)
$N \in V$
$\mathfrak{N} := (N, \in)$. Often confuse N with $\mathfrak{N}$

Fact (scheme): for fixed metamathematical n and any set X, there exists $\mathfrak{N} \prec_{\Sigma_n} (V, \in)$ with $X \subset \mathfrak{N}$ and $|\mathfrak{N}| = |X| + \aleph_0$.

Often omit n and write $\mathfrak{N} \prec_* (V, \in)$

Suppose $\mathfrak{N} \prec_* (V, \in)$. Then

1. ω is element and subset of $\mathfrak{N}$
   - Cor: if $z \in \mathfrak{N}$ and z is countable, then $z \subset \mathfrak{N}$ (why?)
   - converse is false! Ctble subsets of $\mathfrak{N}$ often fail to be elements of $\mathfrak{N}$!
   - but **finite** subsets/tuples of $\mathfrak{N}$ are elements of $\mathfrak{N}$. **Cor**: for finitary 1st order signature $\mathcal{L}$,

$$\mathcal{L} \cup \{\mathcal{L}\} \subset \mathfrak{N} \prec_* (V, \in) \implies \forall A \in \mathfrak{N} \cap \mathrm{Str}_{\mathcal{L}} \ \ \mathfrak{N} \cap A \prec A.$$

Alan Dow [6]: set-theoretic elementary submodel arguments provide

1. *a convenient shorthand encompassing all standard closing-off arguments;*
2. *a powerful technical tool which can be avoided but often at great cost in both elegance and clarity . . .*

Submodules of free modules

F: a free R-module (has an R-basis)
A: a submodule of F

1. A can fail to be free: $R = F = \mathbb{Z}/4\mathbb{Z}$, $A = 2\mathbb{Z}/4\mathbb{Z}$
2. F/A can fail to be free: $R = \mathbb{Z}$, $F = \mathbb{Z}$, $A = 2\mathbb{Z}$

There are also examples even when A is an elementary submodule of F in the language of R-modules

Some **very** nicely behaved submodules

Lemma

Suppose F is a free R-module. Then

$$\{R, F\} \subset \mathfrak{N} \prec_* (V, \in) \implies \langle \mathfrak{N} \cap F \rangle \text{ and } F/\langle \mathfrak{N} \cap F \rangle \text{ are free}$$

Notice: R *and* F *are elements of, not necessarily subsets of,* $\mathfrak{N}$.
"a.e.$\mathfrak{N}$ $\langle \mathfrak{N} \cap F \rangle$ *and* $F/\langle \mathfrak{N} \cap F \rangle$ *are free"*

Why are $\langle \mathfrak{N} \cap F \rangle$ and $F / \langle \mathfrak{N} \cap F \rangle$ free?

1. $(V, \in) \models$ " $\exists B \; B$ is an R-basis of F"
2. Since $F, R \in \mathfrak{N} \prec_* (V, \in)$, there is such a $\mathbf{B} \in \mathfrak{N}$.
3. $F = \langle B \rangle = \underbrace{\langle \mathfrak{N} \cap B \rangle}_{\text{free}} \oplus \underbrace{\langle B \setminus \mathfrak{N} \rangle}_{\text{free}}$
4. So it suffices to show $\langle \mathfrak{N} \cap B \rangle = \langle \mathfrak{N} \cap F \rangle$.
5. Nontrivial direction: assume $x \in \langle \mathfrak{N} \cap F \rangle$. So $x \in \langle x_1, \ldots, x_\ell \rangle$ where each $x_i \in \mathfrak{N} \cap F$.
   - Consider a fixed $x_i \in \mathfrak{N} \cap F$.
     $(V, \in) \models (\exists b_1 \in \mathbf{B} \ldots \exists b_k \in \mathbf{B}) \;\; \mathbf{x_i} \in \langle b_1, \ldots, b_k \rangle^{\mathbf{F}}_{\mathbf{R}}$.
   - Boldface parameters are elements of $\mathfrak{N}$, which is elementary in $(V, \in)$. So the b_j are in $\mathfrak{N} \cap B$. So $x_i \in \langle \mathfrak{N} \cap B \rangle$.
6. So $x \in \langle \mathfrak{N} \cap B \rangle$.

Same for projectives

Corollary

Suppose P is a **projective** *R-module. Then*

$$\{R, P\} \subset \mathfrak{N} \prec_* (V, \in) \implies \langle \mathfrak{N} \cap P \rangle \text{ and } P / \langle \mathfrak{N} \cap P \rangle \text{ are } \textbf{projective}$$

"a.e.$\mathfrak{N}$ $\langle \mathfrak{N} \cap P \rangle$ and $P / \langle \mathfrak{N} \cap P \rangle$ are projective"

$(V, \in) \models$ "$(\exists F\ \exists X)\ F = P \oplus X$ and F is free". Since $R, P \in \mathfrak{N} \prec_* (V, \in)$, can assume F, X are **elements of** $\mathfrak{N}$. Then

$$\underbrace{\langle \mathfrak{N} \cap F \rangle}_{\text{free}} = \langle \mathfrak{N} \cap (P \oplus X) \rangle = \langle \mathfrak{N} \cap P \rangle \oplus \langle \mathfrak{N} \cap X \rangle$$

$$\text{and} \quad \frac{F}{\underbrace{\langle \mathfrak{N} \cap F \rangle}_{\text{free}}} \simeq \frac{P}{\langle \mathfrak{N} \cap P \rangle} \oplus \frac{X}{\langle \mathfrak{N} \cap X \rangle}$$

So $\langle \mathfrak{N} \cap P \rangle$ and $P / \langle \mathfrak{N} \cap P \rangle$ are direct summands of free modules.

Application: reprove Kaplansky's Theorem

Theorem (Kaplansky, 1950s)

Every projective module is a direct sum of countably generated (projective) modules

For simplicity assume R is countable. Assume P is (unctble) projective R-module and the claim holds for all projectives of size $< |P|$.

1. Fix large $V_\alpha \prec_* V$ with $R, P \in V_\alpha$. Set $\mathfrak{A} := (V_\alpha, \in, R, P)$
2. Use Downward L-S to recursively construct $\subseteq$-increasing and continuous chain $\langle \mathfrak{N}_\alpha \ : \ \alpha < \mathrm{cf}(|P|) \rangle$ of elementary submodels of $\mathfrak{A}$, with each $|\mathfrak{N}_\alpha| < |P|$ and $\mathfrak{N}_\alpha \cup \{\mathfrak{N}_\alpha\} \subset \mathfrak{N}_{\alpha+1}$. Then

$$\mathfrak{N}_\alpha \cap P \in \mathfrak{N}_{\alpha+1}$$

3. $Q_\alpha := \frac{\mathfrak{N}_{\alpha+1} \cap P}{\mathfrak{N}_\alpha \cap P} \simeq \mathfrak{N}_{\alpha+1} \cap \frac{P}{\mathfrak{N}_\alpha \cap P}$. This is projective, by previous slide applied twice. So the inclusion splits, and $P \simeq \bigoplus_\alpha Q_\alpha$
4. By IH, each Q_α is direct sum of countably-generated projectives.

Set-theoretic elementary submodels really shine

when objects are more complicated (complexes, accessible categories, ...)

Ex: Suppose

$$\mathbf{A}_\bullet : \qquad \dots A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \dots$$

is an exact complex of abelian groups. Subcomplexes aren't necessarily exact, but if

$$A_\bullet \in \mathfrak{N} \prec_* (V, \in)$$

then

$$\mathbf{A}_\bullet \restriction \mathfrak{N} : \qquad \dots \mathfrak{N} \cap A_{n-1} \xrightarrow{f_{n-1} \restriction \mathfrak{N}} \mathfrak{N} \cap A_n \xrightarrow{f_n \restriction \mathfrak{N}} \mathfrak{N} \cap A_{n+1} \dots$$

is exact! (why?)

Outline

1. Part 1: set-theoretic elementary submodels warmup: Kaplansky's Theorem

2. Part 2: Cellular generation, a characterization, and applications

Cellular generation of a class of morphisms

Fix a class $\mathcal{M}$ of morphisms in a "nice" category containing all isomorphisms.

1. A **transfinite composition of members of $\mathcal{M}$** is a morphism f that can be written as a directed colimit of a wellordered system

$$A_0 \dashrightarrow A_\alpha \xrightarrow{f_{\alpha,\alpha+1}\in\mathcal{M}} A_{\alpha+1} \dashrightarrow A_j \dashrightarrow A_\mu, \qquad f : A_0 \to A_\mu$$

   where directed colimits were taken at all limit j.

2. A map g is a **pushout of a member of $\mathcal{M}$** if it fits in some pushout square like this:

$$\begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ \uparrow & & \uparrow \\ \bullet & \xrightarrow[m\in\mathcal{M}]{} & \bullet \end{array}$$

$\mathcal{M}$ is **cellularly closed** if it is closed under transf comps and pushouts. $\mathcal{M}$ is **cellularly generated** if it is the cellular closure of a **set**.

Example in **Ab**

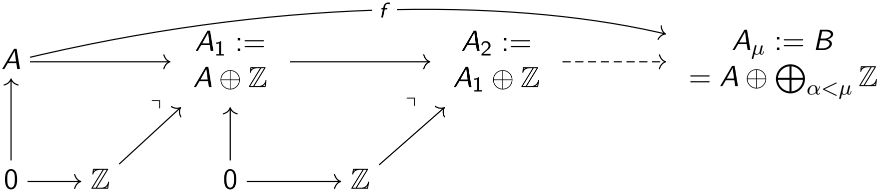
$$\text{Free-Mono} := \{f : A \hookrightarrow B \ : \ B/A \text{ is free}\}$$

is cellularly generated, by the singleton set $\{0 \to \mathbb{Z}\}$.

Say $f : A \subset B$ is in Free-Mono, so

$$B/A \simeq \bigoplus_{\alpha<\mu} \mathbb{Z}$$

for some ordinal μ. Factor f as



Non-examples are harder to get. More on that later.

Significance of Cellular Generation: unlocks **Quillen's Small Object Argument (SOA)**

- If $\mathcal{X}$ is a class of modules, cellular generation of

$$\mathcal{X}\text{-Mono} := \{f \in \text{Mono} \ : \ \text{coker}(f) \in \mathcal{X}\}$$

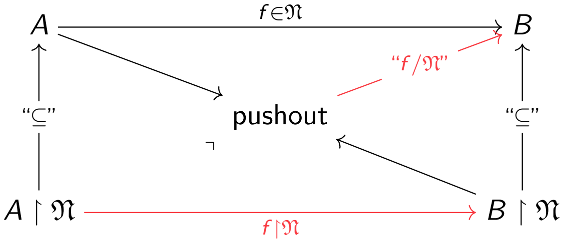
  is basically equivalent to $\mathcal{X}$ being a **deconstructible** class, ensuring (via SOA) existence of $\mathcal{X}$-precovers (Eklof–Trlifaj [7], Rosický [11], and Saorín–Šťovíček [12]).
- Lieberman-Rosický-Vasey [10]: under some technical assumptions on a class $\mathcal{M}$ of morphisms in a category $\mathcal{K}$:
  - There is at most one stable independence relation on $\Big(\text{Obj}(\mathcal{K}), \mathcal{M}\Big)$: the **$\mathcal{M}$-effective squares**.
  - The $\mathcal{M}$-effective squares is a stable indep. relation **iff** $\mathcal{M}$ is cellularly generated.
- (Quillen) Construction of Model Categories in homotopy theory.

New top-down characterization of cellular generation

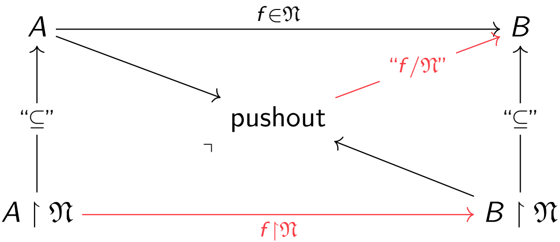
For a locally presentable category $\mathcal{K}$, there is a set $P_{\mathcal{K}}$ such that if $P_{\mathcal{K}} \subset \mathfrak{N} \prec_* (V, \in)$, "$f \restriction \mathfrak{N}$" makes sense for any morphism $f \in \mathfrak{N}$.

E.g., for $\mathcal{K} = R$-Mod, $P_{\mathcal{K}} = R \cup \{R\}$.

For $\mathcal{K} = \mathrm{Mod}(T)$, $P_{\mathcal{K}} = \mathcal{L} \cup T \cup \{\mathcal{L}, T\}$.



New top-down characterization of cellular generation



Theorem (C. [4] for monoid acts; C.-Kamsma-Rosický (forthcoming) for general case): Let $\mathcal{M}$ be a cellularly-closed class of morphisms in a locally presentable category $\mathcal{K}$. The following are equivalent:

1. $\mathcal{M}$ is cellularly generated
2. $\mathcal{M}$ is **a.e. quasi-effective**, meaning there is a set $P \supseteq P_{\mathcal{K}}$ such that whenever $P \subset \mathfrak{N} \prec_* (V, \in)$:

$$f \in \mathfrak{N} \cap \mathcal{M} \implies (f \restriction \mathfrak{N} \text{ and } f/\mathfrak{N} \text{ are both in } \mathcal{M})$$

"a.e.$\mathfrak{N}$ $\forall f \in \mathfrak{N} \left(f \in \mathcal{M} \implies f \restriction \mathfrak{N} \text{ and } f/\mathfrak{N} \text{ are both in } \mathcal{M} \right)$."

Important special case of the characterization is the older:

Theorem (C. [3], [2])

A class $\mathcal{C}$ of R-modules is deconstructible if and only if

$$a.e.\mathfrak{N}\Big(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } C/(\mathfrak{N} \cap C) \in \mathcal{C}\Big)$$

Will not prove the characterizations here. Focus on applications of them.

Intersection of set-many cellularly generated classes

Theorem

(Stovicek?) Suppose I is a set and $\mathcal{M}_i$ is cellularly generated for each $i \in I$. Then $\bigcap_{i \in I} \mathcal{M}_i$ is cellularly generated.

By the new characterization, for each i there is a set P_i such that

$$P_i \subset \mathfrak{N} \prec_* (V, \in) \implies (\forall f \in \mathfrak{N} \cap \mathcal{M}_i)(f \restriction \mathfrak{N} \text{ and } f/\mathfrak{N} \text{ are in } \mathcal{M}_i).$$

Set $P := \bigcup_{i \in I} P_i$, let $f \in \bigcap_i \mathcal{M}_i$, and suppose $P \subset \mathfrak{N} \prec_* (V, \in)$. Then for each i, $P_i \subset \mathfrak{N}$, so both $f \restriction \mathfrak{N}$ and $f/\mathfrak{N}$ are in $\mathcal{M}_i$.

So by the new characterization, P witnesses $\bigcap_{i \in I} \mathcal{M}_i$ is cellularly generated.

(brushing aside some technical metamathematical issues here)

Flat Cover Conjecture for R-Mod

Theorem (Eklof-Trlifaj, Bican-El Bashir-Enochs)

For any ring R, every R-module has an $\mathcal{F}_R$-cover, where $\mathcal{F}_R$ is the class of flat modules.

3 parts:

1. Suffices to get $\mathcal{F}_R$-precovers (Enochs)
2. to get precovers, suffices to show $\mathcal{F}$ is a deconstructible class (Eklof-Trlifaj SOA).
3. (Enochs) Show that $\mathcal{F}$ is deconstructible; equivalently, that $\mathcal{F}$-Mono is cellularly generated.

We'll prove the last part

Prove $\mathcal{F}$ (=flat modules) is deconstructible

By the special case (C. [3]) of the characterization, suffices to show that if

$$R \cup \{R\} \subset \mathfrak{N} \prec_* (V, \in)$$

and $F \in \mathfrak{N}$ is flat, then so are $\mathfrak{N} \cap F$ and $F/(\mathfrak{N} \cap F)$.

Since R is both an element and subset of $\mathfrak{N}$, $\mathfrak{N} \cap F$ is elementary in F in language of R-modules. In particular, it's pure in F and so

$$0 \to \mathfrak{N} \cap F \to F \to F/(\mathfrak{N} \cap F) \to 0$$

is a pure exact sequence with flat middle term.

It follows (Lam [8], Cor 4.86) that the other 2 nonzero terms are flat.

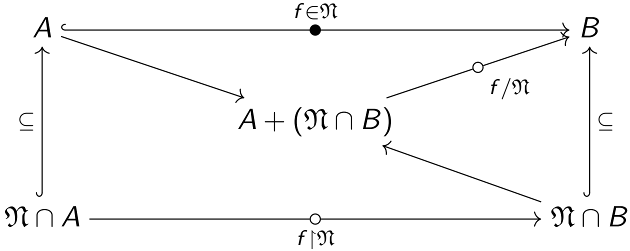
Thm (C. [4]) The FCC also holds for Act-S (a non-additive category) under certain conditions on the monoid S

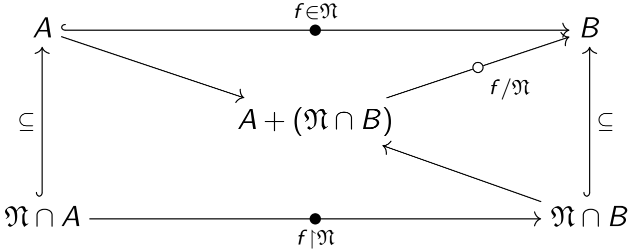
Pure monos cofibrantly generated in R-Mod

(originally Lieberman et al. [9])

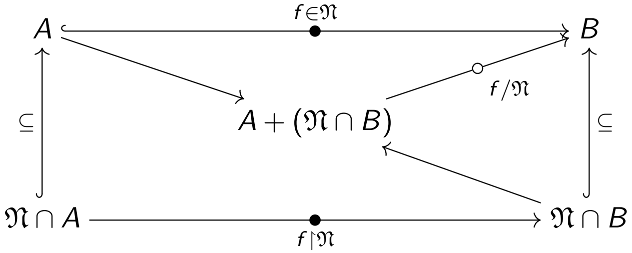
By new characterization, **suffices to show** that if $f : A \subset B$ is pure then

$$R \cup \{R, f\} \subset \mathfrak{N} \prec_* (V, \in) \implies f \restriction \mathfrak{N} \text{ and } f/\mathfrak{N} \text{ are pure.}$$





Still need to show $f/\mathfrak{N}$ is pure.



Key point: since $A, B \in \mathfrak{N}$, so is B/A; and:

- $\mathfrak{N} \cap (B/A)$ is pure (in fact elementary) in B/A

Then for any finitely-presented module X,

$$\mathrm{Hom}(X, B) \twoheadrightarrow \mathrm{Hom}(X, B/A) \twoheadrightarrow \mathrm{Hom}\Big(X, \underbrace{(B/A)/(\mathfrak{N} \cap (B/A))}_{\simeq B/(A+(\mathfrak{N}\cap B))}\Big)$$

When are pure monos cellularly generated in monoid acts?

S: a monoid
S-Act: category of (left) actions of S (non-additive analogue of R-Mod)

Theorem (C.-Feigert-Kamsma-Mazari Armida-Rosický [5])

Pure monos in are cellularly generated the category S-Act if and only if S is (left) LO.

(i.e., iff $\forall s, t \in S$: t is a mult of s or vice versa)
E.g. $(\mathbb{N}, 0, +)$ is LO, $(\mathbb{N}, 1, \cdot)$ is not.

$\Leftarrow$ proved earlier by Borceaux-Rosický.

We'll focus on $\Rightarrow$. We'll do the proof that was ultimately replaced by model-theoretic ones in [5].

Suppose s, t witness non-LO of S and suppose, toward a contradiction, that the pure monos are cellularly generated in S-Act. Hence, the pure monos are a.e. quasieffective, i.e. there is a set P s.t.

$$P \subset \mathfrak{N} \prec_* (V, \in) \implies \forall f \in \mathfrak{N}\Big(f \text{ pure} \quad \implies \quad f \restriction \mathfrak{N} \text{ and } f/\mathfrak{N} \text{ are pure}\Big)$$

Fix disjoint sets A, B each of size $\geq |P|^{++}$. **Fact:** since neither s nor t is a multiple of the other, there is a $Y \supseteq A \sqcup B$ and an S-act on Y such that

$$\forall a \in A \, \forall b \in B \, \exists y_{a,b} \in Y \;\, sy_{a,b} = a \text{ and } ty_{a,b} = b.$$

(see board)

Recall $|P|^{++} \leq \min(|A|, |B|)$. Use LS to sequentially pick two elementary submodels $\mathfrak{M}$ and then $\mathfrak{N}$, both containing $P \cup S$ (as a subset) and A, B, Y, S, s, t (as elements) such that:

- $|\mathfrak{M}| = |P|^+ \subset \mathfrak{M}$.
- $\mathfrak{M} \in \mathfrak{N}$ and $|\mathfrak{N}| = |P| \subset \mathfrak{N}$.

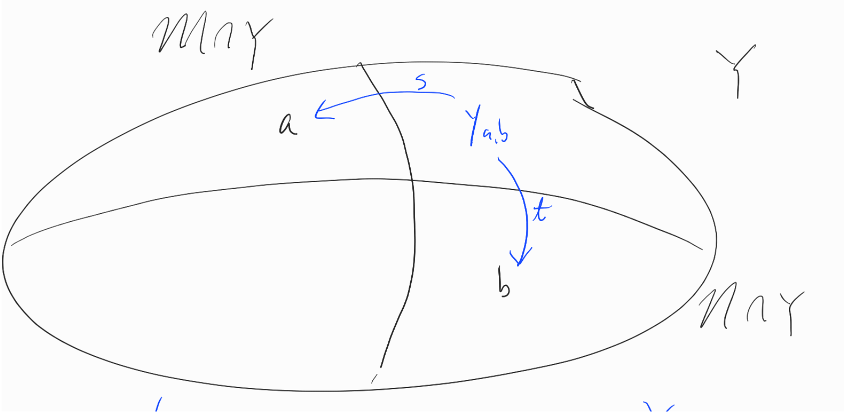
These imply in particular that

$$(\mathfrak{M} \cap A) \setminus \mathfrak{N} \text{ is nonempty}$$

and

$$(\mathfrak{N} \cap B) \setminus \mathfrak{M} \text{ is nonempty}$$

(why?)
Fix $a \in (\mathfrak{M} \cap A) \setminus \mathfrak{N}$ and $b \in (\mathfrak{N} \cap B) \setminus \mathfrak{M}$. Then there is $y_{a,b}$ with $sy_{a,b} = a$ and $ty_{a,b} = b$.

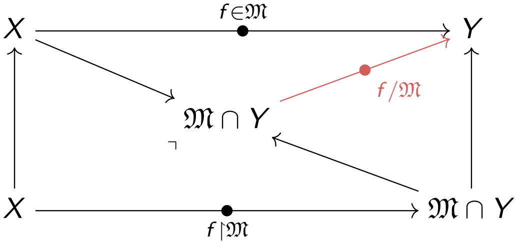


So the equations $sy = a$, $ty = b$ in variable y are satisfied, and the parameters a, b come from the union $(\mathfrak{M} \cap Y) \cup (\mathfrak{N} \cap Y)$. **If we can prove that**

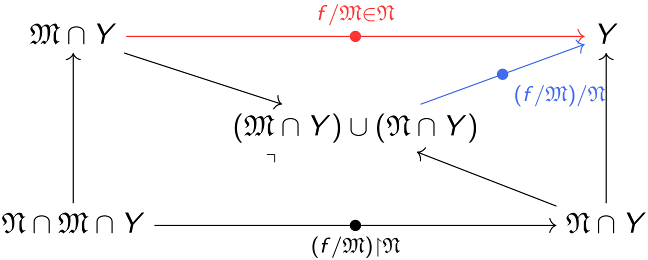
$$(\mathfrak{M} \cap Y) \cup (\mathfrak{N} \cap Y) \text{ is pure in } Y, \tag{1}$$

we'll have a contradiction. (why?)
Now we prove (1).

Fix any $X \in \mathfrak{M} \cap \mathfrak{N}$ that is pure in Y and of size $\leq |P|$. Then $X \subset \mathfrak{M}$ and (dots indicate pure inclusions):



Since $\mathfrak{M}, X, Y \in \mathfrak{N}$, $f/\mathfrak{M} \in \mathfrak{N}$ and is pure. Then:



Gorenstein projective precovers, and club-determinacy

An R-module G is Gorenstein Projective if there is an exact complex $P_\bullet$ of projectives with $G = \ker(P_0 \to P_1)$ and $P_\bullet$ is $\mathrm{Hom}(-, \mathrm{Proj})$-exact.
A dichotomy:

Theorem (C. [3])

If $\kappa > |R|$ is regular then for any G.P. module G,

$$\{A \in [G]^{<\kappa} \ : \ A \text{ is G.P}\}$$

either contains a club, or is disjoint from one.

Corollary

(C. [3], indep Cortes-Izurdiaga and Šaroch [1]) If $\kappa > |R|$ is supercompact, GP is a deconstructible class.

Q: Can GP consistently fail to be deconstructible for some ring? Does the "disjoint from a club" part ever hold (over some ring) for all κ?

[1] Manuel Cortés-Izurdiaga and Jan Šaroch, *The cotorsion pair generated by the Gorenstein projective modules and λ-pure-injective modules*, Israel J. Math. (2025), 1–30.

[2] Sean Cox, *Approximation Theory and Elementary Submodels*, arXiv preprint https://arxiv.org/abs/2405.19634.

[3] ———, *Maximum deconstructibility in module categories*, Journal of Pure and Applied Algebra **226** (2022), no. 5.

[4] ———, *The flat cover conjecture for monoid acts*, J. Lond. Math. Soc. (2) **112** (2025), no. 6, Paper No. e70404.

[5] Sean Cox, Jonathan Feigert, Mark Kamsma, Marcos Mazari-Armida, and Jiří Rosický, *Cofibrant generation of pure monomorphisms in presheaf categories*, Advances in Mathematics **497** (2026), Article 111010, DOI 10.1016/j.aim.2026.111010.

[6] Alan Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. **13** (1988), no. 1, 17–72. MR1031969 (91a:54003)

[7] Paul C. Eklof and Jan Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), no. 1, 41–51.

[8] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR1653294

[9] Michael Lieberman, Leonid Positselski, Jiří Rosický, and Sebastien Vasey, *Cofibrant generation of pure monomorphisms*, J. Algebra **560** (2020), 1297–1310.

[10] Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Cellular categories and stable independence*, J. Symb. Log. **88** (2023), no. 2, 811–834.

[11] J. Rosický, *Flat covers and factorizations*, J. Algebra **253** (2002), no. 1, 1–13.

[12] Manuel Saorín and Jan Šťovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. **228** (2011), no. 2, 968–1007.