QUALIFYING EXAM: TOPOLOGY May 2023

PART I

Complete the following definitions/theorem statements in the space provided. (Fill in any blanks and complete any phrases indicated by a trailing …)

Problem 1: Let A be a set in a topological space X. A point $a \in X$ is a *limit point of* A provided that …

Problem 2: Let $\sim$ an equivalence relation on the topological space X. The *quotient space* $X/\sim$ is …

Problem 3: A topological space X is *metrizable* if …

Problem 4: A topological space X is *separable* provided that …

Problem 5: Given paths $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ in X with $\alpha(1) = \beta(0)$, then $\alpha * \beta$ is …

Qualifying Exam: Topology May 2023

PART I (continued)

Complete the following definitions/theorem statements in the space provided. (Fill in any blanks and complete any phrases indicated by a trailing …)

Problem 6: Let $\sigma = \{a_0, \ldots, a_p\}$ be an oriented p-simplex. Then $\partial\sigma$ is …

Problem 7: Let X be a topological space and ≥ 0. The p-th singular chain group $S_p(X)$ is …

Problem 8: The Excision Axiom for a homology theory is the following: "If (X, A) is admissible and …

Problem 9: Suppose that $\{X_1, X_2\}$ is an excissive couple for the space X. The associated Mayer-Vietoris sequence is given by …

Problem 10: Let G be an abelian group and $\mathcal{C}$ a chain complex. The cochain complex of $\mathcal{C}$ with G coefficients is given by …

QUALIFYING EXAM: TOPOLOGY May 2023

PART II

Complete EXACTLY ONE problem from EACH of the following groups of problems, each on their own page(s). Indicate the number of the problem on each page.

STAPLE your pages to PART 1 of this exam when complete.

Problem 1a : Let X be a compact Hausdorff space. Prove that X is normal.

Problem 1b : Let X and Y be topological spaces and $f : X \to Y$ a function. Prove that the following are equivalent:

(i) f is continuous.

(ii) For each $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

(iii) For each $B \subseteq Y$ which is closed, $f^{-1}(B)$ is closed in X.

Problem 2a : Let X be a space such that for every collection $\mathcal{C}$ of closed sets with the finite intersection property, the intersection $\bigcap \mathcal{C}$ is nonempty. Prove that X is compact.

Problem 2b : Prove the Heine-Borel Theorem in $\mathbb{R}^1$, i.e. prove that a subspace of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Problem 3a : Let X be a topological space and A a subspace of X. Prove that if A is a deformation retract of X, then the inclusion map from A to X induces an isomorphism of fundamental groups.

Problem 3b : Prove that if $f : S^1 \to S^1$ is nulhomotopic (and continuous), then there exists $x \in S^1$ such that $f(x) = -x$

Problem 4a : Let K and L be simplicial complexes and $f, g : K \to L$ be contiguous simplicial maps. Prove that the induced homology maps f_* and g_* are equal.

Problem 4b : Fix $n > 0$ and let K be the simplicial complex consisting of an n-simplex and all of its faces. Prove that $|K|$ is acyclic.

Problem 5a : Let $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ be a short exact sequence of chain complexes. Prove that the long sequence of homology groups

$$\cdots \to H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \to \cdots$$

is exact. [You need not prove that ∂_* is a well-defined homomorphism]

Problem 5b : Let $n > 0$ Let C be a subset of $\mathbb{S}^n$ homeomorphic to the $n-1$ sphere. Then $\mathbb{S}^n - C$ has precisely two components.

QUALIFYING EXAM: TOPOLOGY May 2023

PART III

Complete EXACTLY FOUR of the following, each on their own page(s).Indicate the number of the problem on each page.

STAPLE your pages to PART 1 of this exam when complete.

Problem 1: Let $\{Y_\alpha : \alpha \in J\}$ be an indexed collection of non-empty topological spaces and let $Y = \prod_{\alpha \in J} Y_\alpha$, equipped with the product topology.

(i) Prove that Y is connected if and only if Y_α is connected for each α. (You may assume that finite products of connected spaces are connected).

(ii) Prove that $f : Y \to Y$ is continuous if and only if $\pi_\beta \circ f : Y \to Y_\beta$ is continuous for each $\beta \in J$ (π_β denotes the projection into the β-th space, i.e. $(y_\alpha)_{\alpha \in J} \mapsto y_\beta$).

Problem 2:

(i) Prove the following result concerning lifts and covering maps (you do not need to show that the lift $\tilde{f}$ exists):

Theorem. *Let $p : E \to B$ be a convering map with $p(e_0) = b_0$. Then $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ given by $\phi([f]) = \tilde{f}(1)$ (where $\tilde{f}$ denotes the lift of f which satisfies $\tilde{f}(0) = e_0$) is well-defined. If E is path-connected, then ϕ is surjective. If E is simply connected, then ϕ is bijective.*

(ii) Use the above result to prove that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$.

Problem 3: Fix $n, m \geq 0$

(i) Prove that $\tilde{H}_p(S^n)$ is isomorphic to $\mathbb{Z}$ if $p = n$ and is trivial if $p \neq n$.

(ii) Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

Problem 4: Find the fundamental groups and homology groups of the following spaces. using ANY appropriate method. ***Briefly*** explain your reasoning.

(i) The 5-fold dunce cap

(ii) The Möbius strip

(iii) The torus

(iv) The torus with a single point removed

Problem 5: Let $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \to \mathcal{E} \to 0$ be a short exact sequence of free chain complexes. Prove that if ϕ induces homology isomorphisms in all dimensions, then it induces cohomology isomorphisms in all dimensions.
