1. (3 points each) Complete each of the following definitions carefully:
   (a) Suppose $G = \{X_a : a \in A\}$ is a collection of sets. The *Cartesian product* ...
   (b) A topological space $(X, \mathcal{T})$ is *normal* .... It is a $T_4$ space ...
   (c) A set $S$ is said to have the *Lindelöff property* ...
   (d) Suppose $f$ and $f'$ are paths in a space $X$. Then $f$ is said to be *path homotopic* to $f'$ ...
   (e) Suppose $X$ is a space and $x_0$, a point of $X$. The *fundamental group* ...

2. (3 points each) Give an example of each of the following or state that no such example exists. You need not show any work.
   (a) a topological space $X$ and a sequence in $X$ that converges to more than one point.
   (b) a $T_1$ space that is not Hausdorff
   (c) a countable basis for the usual topology on $\mathbb{R}^2$.
   (d) a metric space that is not $T_4$
   (e) a topological space $X$ and two points $x_0$ and $x_1$ of $X$ such that $\Pi_1(X, x_0)$ and $\Pi_1(X, x_1)$ are not isomorphic.
   (f) two topological spaces $X$ and $Y$ with points $x_0 \in X$ and $y_0 \in Y$ such that $\Pi_1(X, x_0)$ $\Pi_1(Y, y_0)$ are isomorphic, but $X$ and $y$ are not homeomorphic.
   (g) a retraction of $B^2$ onto $S^1$

3. (10 points each) Prove four of the following:
   (a) Suppose $G = \{X_a : a \in A\}$ is a collection of topological spaces, $X$ is a topological space, and $f$ is a function from $X$ into $\prod_A X_a$. If $\pi_a \circ f$ is continuous for each $a$ in $A$, then $f$ is continuous.
   (b) Every subset of a second countable space $X$ has the Lindelöf property.
   (c) Suppose $G = \{X_a : a \in A\}$ is a collection of topological spaces. Then $\prod_A X_a$ is compact if and only if $X_a$ is compact for each $a$ in $A$.
   (d) Suppose $(X, \mathcal{T})$ is a topological space. If $A$ is a connected subset of $X$, and $A \subset B \subset \overline{A}$ then $B$ is connected.
   (e) Suppose $h : (X, x_0) \to (Y, y_0)$ is a map. Then $h_* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$, given by $h_*([f]) = [h \circ f]$, is a homomorphism.

4. (3 points each) Complete each of the following definitions carefully:

   (a) Let $X$ be a topological space. The $k$-th *singular homology group* $H_k(X)$ is ...
   (As part of your answer also define the group $C_k(X)$ of singular $k$-chains and the boundary operator $\partial_k : C_k(X) \to C_{k-1}(X)$.)

   (b) Let $X$ be a topological space. The 0-th *reduced homology group* $\widetilde{H}_0(X)$ is ...

   (c) A pair $(X, A)$ of topological spaces is called a *good pair* if ...

   (d) Let $f : S^n \to S^n$ be a continuous map. The *degree* of $f$ is ...

   (e) Let $X$ be a CW complex. The $k$-th *cellular homology group* $H_k^{\mathrm{CW}}(X)$ is ...

5. (10 points each) Complete three of the following:

   (a) Let $f, g : X \to Y$ be two continuous maps and let $f_*, g_* : H_k(X) \to H_k(Y)$ be the induced homomorphisms between homology groups. Sketch a proof that if $f \simeq g$, then $f_* = g_*$.

   (b) Prove that if $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then $n = m$.

   (c) Prove Brouwer's fixed-point theorem: Every continuous map $f : D^n \to D^n$ has a fixed point.

   (d) Prove the Borsuk–Ulam theorem: For every continuous map $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$. (You may take for granted that an odd map from a sphere to itself must have odd degree.)

   (e) Let $X$ be a (finite) CW-complex. Prove that if $X$ has no two cells in adjacent dimensions, then $H_k^{\mathrm{CW}}(X)$ is a free abelian group with a basis in one-to-one correspondence with the $k$-cells of $X$.

6. (10 points each) Complete three of the following:

   (a) Let $K^2$ be the Klein bottle. Use the method of your choice to show that

$$H_k(K^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

   (b) Use a Meyer–Vietoris sequence to compute the homology groups $H_k(T^2)$, where $T^2$ is the two-dimensional torus.

   (c) Use the universal coefficient theorems to compute the groups $H_k(K^2; G)$ and $H^k(K^2; G)$, where $G$ is an abelian group.

   (d) Use the standard CW complex structure of $\mathbb{R}P^n$ to compute the cellular homology groups $H_k^{\mathrm{CW}}(\mathbb{R}P^n)$.

Useful facts for computing homology groups with coefficients in $G$

- $\_ \otimes G$ and $\mathrm{Tor}_1(\_, G)$ are additive functors
- $\mathbb{Z} \otimes G = G$
- $(\mathbb{Z}/n\mathbb{Z}) \otimes G = G/nG$
- $\mathrm{Tor}_1(\mathbb{Z}, G) = 0$
- $\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, G) = \{g \in G \mid ng = 0\}$

Useful facts for computing cohomology groups with coefficients in $G$

- $\mathrm{Hom}(\_, G)$ and $\mathrm{Ext}^1(\_, G)$ are additive functors
- $\mathrm{Hom}(\mathbb{Z}, G) = G$
- $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G) = \{g \in G \mid ng = 0\}$
- $\mathrm{Ext}^1(\mathbb{Z}, G) = 0$
- $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$