A New Brand of Hardy Spaces and the Neumann Problem in $\mathcal{UR}$ Domains

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Joint work with Marius Mitrea

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Recent Advances in Harmonic Analysis and Partial Differential Equations

Celebrating Marius Mitrea's career

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Outline

The Neumann Problem on the Hardy Space H^1

Let $\Omega \subseteq \mathbb{R}^n$ be an open set (+ geometric conditions) with surface measure σ on $\partial\Omega$, and outward unit normal ν. Fix an aperture parameter $\kappa \in (0, \infty)$. Consider the Neumann Problem for the Laplacian with data in the Hardy space $H^1(\partial\Omega, \sigma)$:

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^1(\partial\Omega, \sigma), \\ \partial_\nu u := \nu \cdot (\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in H^1(\partial\Omega, \sigma). \end{cases}$$

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Here, $\Gamma_\kappa(x) := \{y \in \Omega : |y - x| < (1 + \kappa)\,\mathrm{dist}(y, \partial\Omega)\}$ denotes the nontangential approach region with vertex at $x \in \partial\Omega$.

Given a $\mathcal{L}^n$-measurable function u defined in Ω, and a point $x \in \partial\Omega$, set

$$(\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^n)} \quad \text{and} \quad \Big(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) := \lim_{\Gamma_\kappa(x) \ni y \to x} u(y).$$

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Note: δ-AR domains is the sharp version, from a GMT point of view, of the class of Lipschitz domains with small Lipschitz constants (cf. [GHA]).

Other Related Works

Related works regarding the Neumann Problem on the Hardy space $H^p(\partial\Omega, \sigma)$ include:

- **[R. Brown, 1995]** in (starlike) Lipschitz domains, in dimension $n \geq 3$.
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- **[D. Mitrea, 2002]** in Lipschitz domains, in dimension $n = 2$.
  - ⋄ For $\frac{2}{3} - \varepsilon < p \leq 1$ with $\varepsilon \in (0, \frac{1}{6}]$ small.

Recent Work

[M. Mitrea, P.T. 2024] Consider an AR domain $\Omega \subseteq \mathbb{R}^n$ with GMT outward unit normal ν. Let $L := A_{jk}\partial_j\partial_k$ be an $M \times M$ weakly elliptic system in $\mathbb{R}^n$, and assume $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$. Fix $p \in (1, \infty)$. Then there exists $\delta \in (0, 1)$ such that whenever $\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} < \delta$ the Neumann Problem with data prescribed in the Beurling-Hardy space is well posed:

$$(\mathrm{HA}^p\text{-NBVP}) \begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \quad Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in \mathrm{A}^p(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \left[\mathrm{HA}^p(\partial\Omega, \sigma)\right]^M. \end{cases}$$

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Fatou-type result: HA^p is the "correct" space of boundary data in the formulation of this Neumann Problem.

L^p-Based Beurling Algebras

Let $\Sigma \subseteq \mathbb{R}^n$ be an unbounded Ahlfors regular set, and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Denote by $\Delta(x, r)$ the surface ball $B(x, r) \cap \Sigma$ on Σ, for each $x \in \Sigma$ and $r > 0$. Fix a reference point $x_0 \in \Sigma$, and $2_* \gg 1$. Set $C_0 := \Delta(x_0, 2_*)$ and $C_k := \Delta(x_0, 2_*^{k+1}) \backslash \Delta(x_0, 2_*^k)$ for each $k \in \mathbb{N}$. Fix $p \in (1, \infty)$.

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Note: The space A^p is quite different (in nature) from L^1 and L^p.

Beurling-Hardy Spaces

Fix a background parameter $\gamma \in (0,1)$. Following [MiTa24], for any $p \in (1,\infty)$, introduce the Beurling-Hardy space by setting

$$\mathrm{HA}^p(\Sigma,\sigma) := \left\{ f \in L^1_{\mathrm{loc}}(\Sigma,\sigma) \,:\, f^{\#}_{\gamma} \in \mathrm{A}^p(\Sigma,\sigma) \right\}$$

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$$f^{\#}_{\gamma}(x) := \sup_{\psi \in \mathcal{B}^{\gamma}_{x}(\Sigma)} |\langle f, \psi \rangle| \quad \forall\, x \in \Sigma,$$

with $\mathcal{B}^{\gamma}_{x}(\Sigma) \subseteq \dot{\mathscr{C}}^{\gamma}(\Sigma)$ being a collection of suitably normalized "bump" functions centered at $x \in \Sigma$.

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Note: HA^p can be thought as a “L^p-flavored” H^1.

Atomic Characterization of HA^p

In the same context as before, a σ-measurable function $a : \Sigma \to \mathbb{R}$ is called an x_0-central L^p-atom provided there exists $R \geq 2_*$ such that

- *Localization:* $\operatorname{supp} a \subseteq \Delta(x_0, R)$,
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These atoms are the building blocks for HA^p. In fact, given any $f \in L^1(\Sigma, \sigma)$, one has

$$f \in \mathrm{HA}^p(\Sigma, \sigma) \iff f = \sum_{k=0}^{\infty} \lambda_k a_k \text{ in } L^1(\Sigma, \sigma)$$

for some sequence of x_0-central L^p-atoms $\{a_k\}_{k \in \mathbb{N}_0}$, and some sequence $\{\lambda_k\}_{k \in \mathbb{N}_0} \in \ell^1(\mathbb{R})$. This characterization is quantitative!

Role of L^p

Recall that the A^p-norm of a σ-measurable function f on Σ is defined as

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  - ⋄ For one thing, L^p is good for doing Harmonic Analysis, $1 < p < \infty$.
  - ⋄ The atomic theory for HA^p is *tied up* with the manner in which the space A^p is "normalized". *This suggests how one can transition from L^p to a generic space $\mathbb{X}$ of σ-measurable functions on Σ.*

In an Abstract Setting

$\mathbb{X} \in ????$ *such that*
$M_{\partial\Omega}$ *is bounded on* $\mathbb{X}$ *and* $\mathbb{X}'$

Connection with Harmonic Analysis

[*Beurling – Hardy Space Template*]

New Hardy Spaces $\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)$
(*of Beurling nature*)

[*Atomic & Molecular Theory, CZ Theory*]
+
[*GHA philosophy*]

Well – posedness of the
Neumann Problem on $\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)$??

New Problem

Motivated by the HA^p template, replacing L^p by a generic function space $\mathbb{X}$ leads to functions spaces $\mathrm{A}_{\mathbb{X}}$ and $\mathrm{HA}_{\mathbb{X}}$ (to be made precise later).

New Problem

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Consider a weakly elliptic $M \times M$ system $L := A_{jk}^{\alpha\beta}\partial_j\partial_k$ in $\mathbb{R}^n$. Formulate the following Neumann Problem with data prescribed in the $\mathbb{X}$-based Beurling-Hardy space:

$$
(\mathrm{HA}_{\mathbb{X}}\text{-NBVP}) \begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \quad Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in \mathrm{A}_{\mathbb{X}}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \left[\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)\right]^M. \end{cases}
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Main Goals

• Introduce a new brand of Hardy spaces $\mathrm{HA}_{\mathbb{X}}$ associated with an $\mathbb{X}$-based Beurling space.

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- Develop a Calderón-Zygmund theory for SIOs on $\mathrm{HA}_{\mathbb{X}}$.

- Implement layer potential techniques to tackle the Neumann Problem with boundary data in $\mathrm{HA}_{\mathbb{X}}$.

Crash Course on GBFS

Fix a measure space $(\mathfrak{X}, \mathfrak{M}, \mu)$. Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a normed vector space contained in $\mathscr{M}(\mathfrak{X}, \mu)$, the set of all μ-measurable functions on $\mathfrak{X}$. Following [GHA], call $\mathbb{X}$ a Generalized Banach Function Space (GBFS) on $(\mathfrak{X}, \mu)$ if for all $f, g \in \mathscr{M}(\mathfrak{X}, \mu)$ one has:

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The *associated space* (a.k.a. *Köthe dual*) of $\mathbb{X}$ is defined as

$$\mathbb{X}' := \left\{ g \in \mathscr{M}(\mathfrak{X}, \mu) : \int_{\mathfrak{X}} |fg| \, d\mu < \infty \ \ \forall f \in \mathbb{X} \right\},$$

equipped with the norm $\|g\|_{\mathbb{X}'} := \sup \left\{ \int_{\mathfrak{X}} |fg| \, d\mu : \|f\|_{\mathbb{X}} \leq 1 \right\}$.

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Example 2. Given $p \in (1, \infty)$, and $w \in A_p(\Sigma, \sigma)$, the Muckenhoupt weighted Lebesgue space $L^p(\Sigma, w)$ is a GBFS, and its Köthe dual is $L^{p'}(\Sigma, w')$, where $w' := w^{1-p'} \in A_{p'}(\Sigma, \sigma)$.

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Other function spaces: *Herz spaces, Orlicz spaces, Zygmund spaces, Muckenhoupt weighted Morrey and Block spaces,* ...

GBFS-based Beurling Spaces

Fix an unbounded Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, and let $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Recall that the Hardy-Littlewood maximal operator $\mathcal{M}_\Sigma$ on Σ acts on each $f \in \mathscr{M}(\Sigma, \sigma)$ according to

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Basic Fact: $\mathrm{A}_{\mathbb{X}}(\Sigma, \sigma) \hookrightarrow L^1(\Sigma, \sigma) \cap \mathbb{X}$ continuously.

The Hardy Space Associated with $\mathrm{A}_{\mathbb{X}}$

Let $\mathbb{X}$ be a GBFS on (Σ, σ) and suppose $\mathcal{M}_{\Sigma}$ is bounded on $\mathbb{X}'$. Having fixed $\gamma \in (0,1)$, define the $\mathbb{X}$-based Beurling-Hardy space on Σ as

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Fact: Elements from $\mathrm{HA}_{\mathbb{X}}(\Sigma, \sigma)$ are actually L^1 functions, and in fact $\mathrm{HA}_{\mathbb{X}}(\Sigma, \sigma) \hookrightarrow H^1(\Sigma, \sigma) \cap \mathbb{X}$ continuously.

Atomic Theory on $\mathrm{HA}_{\mathbb{X}}$

Call a σ-measurable function $a : \Sigma \to \mathbb{R}$ an x_0-central $\mathbb{X}$-atom provided there exists $R \geq 2_*$ such that

$$\operatorname{supp} a \subseteq \Delta(x_0, R), \qquad \|a\|_{\mathbb{X}} \leq \frac{\|\mathbb{1}_{\Delta(x_0,R)}\|_{\mathbb{X}}}{\sigma\big(\Delta(x_0, R)\big)}, \qquad \int_{\Sigma} a \, d\sigma = 0.$$

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Proposition 1 (M. Mitrea, P.T.)

Let $\mathbb{X}$ be a GBFS on (Σ, σ) and suppose $\mathcal{M}_\Sigma$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Then for any $f \in L^1(\Sigma, \sigma)$ it follows that, in a quantitative fashion,

$$f \in \mathrm{HA}_{\mathbb{X}}(\Sigma, \sigma) \quad \Longleftrightarrow \quad f = \sum_{k=0}^{\infty} \lambda_k a_k \text{ in } L^1(\Sigma, \sigma)$$

for some sequence of x_0-central $\mathbb{X}$-atoms $\{a_k\}_{k \in \mathbb{N}_0}$, and some sequence $\{\lambda_k\}_{k \in \mathbb{N}_0} \in \ell^1(\mathbb{R})$.

Molecular Theory on $\mathrm{HA}_{\mathbb{X}}$

Fix $\varepsilon \in (0, \infty)$. Call a σ-measurable function $M : \Sigma \to \mathbb{R}$ an x_0-central $(\mathbb{X}, \varepsilon)$-molecule if there exists $R \geq 2_*$ such that for all $k \in \mathbb{N}_0$ one has

$$\|M \cdot \mathbb{1}_{A_k(x_0,R)}\|_{\mathbb{X}} \leq 2_*^{-k(n-1)\varepsilon} \frac{\|\mathbb{1}_{\Delta(x_0,2_*^k R)}\|_{\mathbb{X}}}{\sigma\big(\Delta(x_0, 2_*^k R)\big)} \quad \text{and} \quad \int_\Sigma M \, d\sigma = 0,$$

where $A_0(x_0, R) := \Delta(x_0, R)$ and $A_k(x_0, R) := \Delta(x_0, 2_*^k R) \setminus \Delta(x_0, 2_*^{k-1} R)$ for $k \in \mathbb{N}$.

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Proposition 2 (M. Mitrea, P.T.)

Let $\mathbb{X}$ be a GBFS on (Σ,σ) and suppose $\mathcal{M}_\Sigma$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Fix $\varepsilon \in (0,\infty)$. Then there exists some constant $C \in (0,\infty)$ such that for every x_0-central $(\mathbb{X},\varepsilon)$-molecule M on Σ one has that

$$M \text{ belongs to } \mathrm{HA}_{\mathbb{X}}(\Sigma,\sigma) \text{ and } \|M\|_{\mathrm{HA}_{\mathbb{X}}(\Sigma,\sigma)} \leq C.$$

Nontangential Maximal Function Estimates

From now on, fix a UR-domain $\Omega \subseteq \mathbb{R}^n$ with $\partial\Omega$ unbounded, and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, denote by ν the GMT outward unit normal to Ω.

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Theorem 1 (M. Mitrea, P.T.)

Let $\mathbb{X}$ be a GBFS on $(\partial\Omega, \sigma)$ and suppose $\mathcal{M}_{\partial\Omega}$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Consider a kernel $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ that is odd and positive homogeneous of degree $1-n$. Introduce the boundary-to-domain convolution type SIO $\mathcal{T}$ acting on each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ as

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Then there exists $C \in (0,\infty)$ such that for all $f \in \mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)$ one has

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{\mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)} \leq C\|f\|_{\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)}.$$

Boundedness of SIOs on $\mathrm{HA}_{\mathbb{X}}$

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$$T^\# f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle \nu(x), y - x\rangle k(x-y) f(y)\, d\sigma(y) \quad \text{at } \sigma\text{-a.e. } x \in \partial\Omega.$$

Boundedness of SIOs on $\mathrm{HA}_{\mathbb{X}}$

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$$\|T^\#\|_{\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)\to\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)} \leq C \|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \ln\big(e/\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\big).$$

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  - The atomic and molecular theory of $\mathrm{HA}_{\mathbb{X}}$ plays a crucial role in the proof of the above estimate. In fact, the main idea is to prove that $T^{\#}$ maps central $\mathbb{X}$-atoms into central $(\mathbb{X}, \varepsilon)$-molecules up to a fixed multiple of $\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \ln\left(e/\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\right)$.

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- **Comments about the proof:**
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  - ◇ Characterization of BMO in terms of GBFS: If $\mathbb{X}$ is a GBFS on $(\partial\Omega,\sigma)$ and $\mathcal{M}_{\partial\Omega}$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$, then for every $f \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ one has

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Weakly Elliptic Systems

Let $n \in \mathbb{N}$, with $n \geq 2$, and $M \in \mathbb{N}$. Fix a second-order, homogeneous, constant complex coefficient, weakly elliptic $M \times M$ system of the format

$$L := A_{jk}\partial_j\partial_k \text{ in } \mathbb{R}^n$$

with each $A_{jk} \in \mathbb{C}^{M\times M}$. The weak ellipticity of the system L means that the characteristic matrix

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Fundamental Fact (cf. [Mi18]): Any weakly elliptic system L admits a "nicely behaved" *matrix-valued fundamental solution E_L*.

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For each coefficient tensor $A \in \mathfrak{A}_L$ and $\xi \in \mathbb{R}^n$, introduce the *directional derivative operator* along ξ associated with A as

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$$\partial_\xi^{A^\top} E_{L^\top}(x) = 0 \text{ for all } x, \xi \in \mathbb{R}^n \setminus \{0\} \text{ with } x \cdot \xi = 0.$$

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The (transpose) double layer potential $K_A^{\#}$ associated with $A \in \mathfrak{A}_L$ acts on each $g \in \left[L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\right)\right]^M$, at σ-a.e. $x \in \partial\Omega$, according to

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$A \in \mathfrak{A}_L^{\mathrm{dis}}$ if and only if $K_A^\#$ is a SIO of chord-dot-normal type (cf. [GHA]).

Back to the Neumann Problem

Let $\mathbb{X}$ be a GBFS on $(\partial\Omega, \sigma)$ and suppose $\mathcal{M}_{\partial\Omega}$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Fix a system L as before written for some choice of $A \in \mathfrak{A}_L$. Recall the formulation of the Neumann Problem with boundary data in the $\mathbb{X}$-based Beurling-Hardy space:

$$(\mathrm{HA}_{\mathbb{X}}\text{-NBVP}) \begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \quad Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in \mathrm{A}_{\mathbb{X}}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \left[\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)\right]^M. \end{cases}$$

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Question: Is $\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)$ the correct space of boundary data in the formulation of this problem?

Fatou-type Result

Theorem 3 (M. Mitrea, P.T.)

Suppose $\mathbb{X}$ is a GBFS on $(\partial\Omega, \sigma)$ such that $\mathcal{M}_{\partial\Omega}$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Assume u is a vector-valued function in Ω satisfying

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Then $(\nabla u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$, and for each $A \in \mathfrak{A}_L$ the conormal derivative $\partial_\nu^A u$ belongs to $[\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)]^M$ quantitatively, i.e., there exists $C \in (0, \infty)$, independent of u, such that

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$$\|\partial_\nu^A u\|_{[\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M} \le C\|\mathcal{N}_\kappa(\nabla u)\|_{\mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)}.$$

This ensures that the boundary condition in $(\mathrm{HA}_{\mathbb{X}}\text{-NBVP})$ is meaningfully formulated.

The Method of Layer Potentials

Consider the (modified) single layer potential whose action on each function $g \in \left[L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\right)\right]^M$ is given by

$$\mathscr{S}_{\text{mod}} g(x) := \int_{\partial\Omega} \left\{ E_L(x-y) - E_L(y) \mathbb{1}_{\mathbb{R}^n \setminus B(0,1)}(y) \right\} g(y)\, d\sigma(y) \quad \forall\, x \in \Omega.$$

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Fix some function $g \in [\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)]^M$ (to be determined later). Setting $u := \mathscr{S}_{\rm mod} g$ in Ω ensures

$$u \in [\mathscr{C}^\infty(\Omega)]^M \ \text{ and } \quad Lu = 0 \ \text{ in } \Omega.$$

So, smoothness and PDE conditions in ($\mathrm{HA}_{\mathbb{X}}$-NBVP) are OK!

Moreover, such a choice of u satisfies

$$(\nabla u)(x) = \int_{\partial\Omega} (\nabla E_L)(x-y)g(y)\, d\sigma(y) \quad \forall\, x \in \Omega.$$

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Since ∇E_L is smooth in $\mathbb{R}^n \setminus \{0\}$, odd, and positive homogeneous of degree $1-n$, we may invoke Theorem 1 to ensure that

$$\|\mathcal{N}_\kappa(\nabla u)\|_{\mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)} \leq C \|g\|_{[\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M} < \infty.$$

Hence, $\mathcal{N}_\kappa(\nabla u) \in \mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)$. So, size condition in $(\mathrm{HA}_{\mathbb{X}}\text{-NBVP})$ is OK!

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Hence, $\mathcal{N}_\kappa(\nabla u) \in \mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)$. So, size condition in $(\mathrm{HA}_{\mathbb{X}}\text{-NBVP})$ is OK!

Question: What does it take of g for the function $u = \mathscr{S}_{\mathrm{mod}} g$ to satisfy the boundary condition $\partial_\nu^A u = f$?

Work done in [GHA] implies that for each $g \in \left[L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\right)\right]^M$ one has the jump-relation

$$\partial_\nu^A(\mathscr{S}_{\rm mod} g) = \left(-\tfrac{1}{2}I + K^{\#}_{A^\top}\right)g.$$

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- **Basic Issue:** Is $-\frac{1}{2}I + K^\#_{A^\top}$ invertible on $\left[\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)\right]^M$?

If $A^\top$ is a *distinguished coefficient tensor* for $L^\top$ (i.e., $A^\top \in \mathfrak{A}^{\mathrm{dis}}_{L^\top}$), then $K^{\#}_{A^\top}$ is a SIO of chord-dot-normal type.

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By our Theorem 2, we have that $K^{\#}_{A^\top} : [\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M \to [\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M$ boundedly and there exists $C \in (0,\infty)$ such that

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If $\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)}$ is sufficiently small ("gently sloped" condition), then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on the Banach space $[\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M$ via a Neumann series.

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$$u := \mathscr{S}_{\mathrm{mod}}\Big[\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)^{-1} f\Big] \quad \text{in } \Omega$$

is a solution of ($\mathrm{HA}_{\mathbb{X}}$-NBVP).

Well-posedness of the Neumann Problem

Main Theorem (M. Mitrea, P.T.)

Let $\Omega \subseteq \mathbb{R}^n$ be a AR-domain, and ν its GMT outward unit normal. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Suppose $\mathbb{X}$ is a GBFS on $(\partial\Omega, \sigma)$ such that $\mathcal{M}_{\partial\Omega}$ is bounded both on $\mathbb{X}$ and $\mathbb{X}'$. Consider a weakly elliptic system L in $\mathbb{R}^n$ as before, and assume that $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$.

Then there exists some $\delta \in (0,1)$ with the property that $(\mathrm{HA}_{\mathbb{X}}\text{-NBVP})$ is well posed whenever $\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} < \delta$. Specifically, the operator

$$-\tfrac{1}{2}I + K^{\#}_{A^\top} \text{ is invertible on } [\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M$$

and the function

$$u(x) := \mathscr{S}_{\mathrm{mod}}\Big[\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)^{-1} f\Big](x) \quad \forall\, x \in \Omega,$$

is a solution of $(\mathrm{HA}_{\mathbb{X}}\text{-NBVP})$, which is unique modulo constants, and satisfies $\|\mathcal{N}_\kappa(\nabla u)\|_{\mathrm{A}_{\mathbb{X}}(\partial\Omega,\sigma)} \approx \|f\|_{[\mathrm{HA}_{\mathbb{X}}(\partial\Omega,\sigma)]^M}$.

A GBFS Sampler

Working in the context of our main result, fix $p \in (1, \infty)$, and select a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Choose $\mathbb{X} := L^p(\partial\Omega, w)$. The Neumann Problem with boundary data in the $L^p(\partial\Omega, w)$-based Beurling-Hardy space $\mathrm{HA}^p(\partial\Omega, w) := \mathrm{HA}_{L^p(\partial\Omega,w)}(\partial\Omega, \sigma)$ reads as

A GBFS Sampler

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$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \quad Lu = 0 \text{ in } \Omega, \\ \displaystyle\sum_{k=0}^{\infty} 2_*^{k(n-1)} \Big(⨍_{C_k} \left[\mathcal{N}_\kappa(\nabla u)\right]^p \, dw \Big)^{1/p} < \infty, \\ \partial_\nu^A u = f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{where} \quad \displaystyle\sum_{k=0}^{\infty} 2_*^{k(n-1)} \Big(⨍_{C_k} \left[f_\gamma^\#\right]^p \, dw \Big)^{1/p} < \infty. \end{cases}$$

Big Picture

The theory of $\mathbb{X}$-based Beurling-Hardy spaces gives a recipe for producing new Hardy spaces (of Beurling nature) in which the Neumann BVP is well posed.

$\mathbb{X} \in \mathrm{GBFS}(\partial\Omega, \sigma)$

$\mathcal{M}_{\partial\Omega}$ *is bounded on* $\mathbb{X}$ *and* $\mathbb{X}'$

Connection with Harmonic Analysis

[*Beurling – Hardy Space Template*]

New Hardy Spaces $\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)$

(*of Beurling nature*)

[*Atomic & Molecular Theory, CZ Theory*]

+

[*GHA philosophy*]

Neumann Problem on $\mathrm{HA}_{\mathbb{X}}(\partial\Omega, \sigma)$ *is well posed*

Examples of δ-AR Domains

The "gently sloped" condition captured in the demand $\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} < \delta$ is illustrated by the following examples.

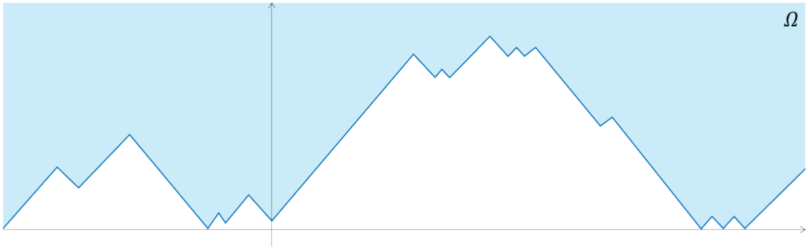

Figure: An upper-graph Lipschitz domain with small Lipschitz constant

Examples of δ-AR Domains

A non-Lipschitz domain for which our theory applies.

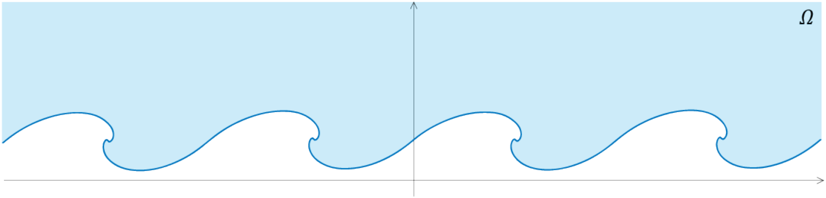

Figure: Example of a δ-AR domain which is not of upper-graph type.

Thank you for your attention!