On the Geometric Harmonic Analysis Philosophy

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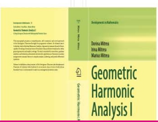
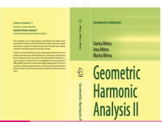
Recent Advances in Harmonic Analysis and PDE

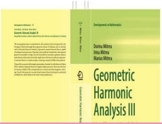
Celebrating Marius Mitrea's Contributions to HA and PDE

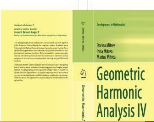
Department of Mathematics, Baylor University

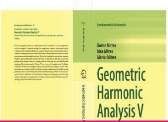
May 19th-22nd, 2025











Geometric Harmonic Analysis

- Geometric Harmonic Analysis I: *A Sharp Divergence Theorem with Nontangential Pointwise Traces*
- Geometric Harmonic Analysis II: *Function Spaces Measuring Size and Smoothness on Rough Sets*
- Geometric Harmonic Analysis III: *Integral Representations, Calderón-Zygmund Theory, Fatou Theorems, and Applications to Scattering*
- Geometric Harmonic Analysis IV: *Boundary Layer Potentials in Uniformly Rectifiable Domains, and Applications to Complex Analysis*
- Geometric Harmonic Analysis V: *Fredholm Theory and Finer Estimates for Integral Operators, with Applications to Boundary Problems*

Background and Motivation: The Case of the Upper-Half Space

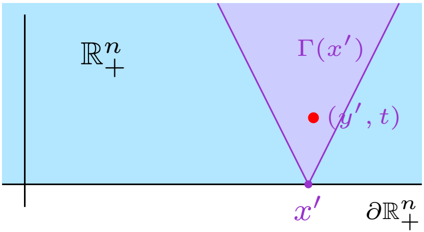
- L^p-Dirichlet BVP, $1 < p < \infty$:

$$\begin{cases} u \in C^\infty(\mathbb{R}^n_+) \\ \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n_+ \\ \mathcal{N}_\kappa u \in L^p(\mathbb{R}^{n-1}) \\ u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \in L^p(\mathbb{R}^{n-1}) \end{cases} \quad \rightsquigarrow$$

Well-posed $\rightsquigarrow \exists\,!$ solution

$$u(x', t) = (P_t^\Delta * f)(x'), \ (x', t) \in \mathbb{R}^n_+$$

$$\|\mathcal{N}_\kappa u\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^{n-1})}$$



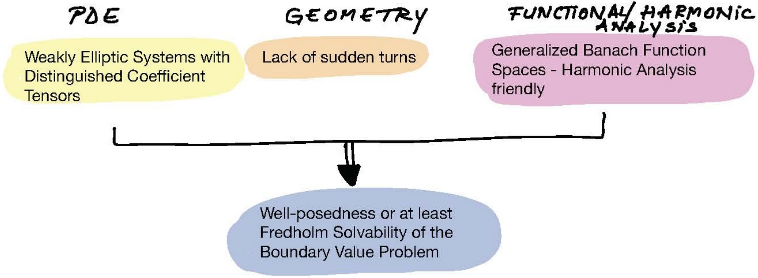
$\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$

- $\mathcal{N}_\kappa u(x') = \sup\limits_{\Gamma_\kappa(x')} |u|, \quad x' \in \mathbb{R}^{n-1}$
- $u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}(x') = \lim\limits_{\substack{(y',t)\to x' \\ (y',t)\in\Gamma_\kappa(x')}} u(y', t)$

GHA Philosophy

Identify structural active ingredients in making an elliptic boundary value problem Fredholm solvable/well-posed.

The Geometric Harmonic Analysis philosophy amounts to:



(1) PDE: $\Delta \rightsquigarrow$ more general second/higher order systems with DCT
(2) Geometry: $\mathbb{R}^n_+ \rightsquigarrow$ domains without sudden big turns
(3) Functional Analysis/Harmonic Analysis: $L^p(\partial\Omega) \rightsquigarrow$ Generalized Banach Function Spaces

(1) PDE - Weakly Elliptic Systems

Fix $n, M \in \mathbb{N}$, with $n \geq 2$ and consider a homogeneous $M \times M$ second-order complex constant coefficient system in $\mathbb{R}^n$

$$L = A_{rs}\partial_r\partial_s \quad \text{with} \quad A_{rs} \in \mathbb{C}^{M\times M}$$

which is weakly elliptic, i.e., its $M \times M$ symbol matrix

$$L(\xi) := -\xi_r\xi_s A_{rs} \quad \text{is non-singular} \quad \forall\, \xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \setminus \{0\}.$$

Examples:

- Scalar operators: $L = A_{rs}\partial_r\partial_s$ with $A_{rs} \in \mathbb{C}$ (e.g., Δ),
- Genuine systems: $L = \mu\Delta + (\lambda + \mu)\nabla\text{div}$ with $\mu, \lambda \in \mathbb{C}$ (Lamé-like).

Special objects for weakly elliptic systems

Given a weakly elliptic system L in $\mathbb{R}^n$ - **very special functions**:

- a fundamental solution $E \in [C^\infty(\mathbb{R}^n \setminus \{0\})]^{M\times M}$ for L. Always exists.
- a Poisson kernel for L in $\mathbb{R}^n_+$. Doesn't always exist.
- a Green function for L in $\mathbb{R}^n_+$. With $x_o \in \mathbb{R}^n$ fixed:
  - $G(\cdot, x_o) \in [L^1_{loc}(\mathbb{R}^n_+)]^{M\times M}$
  - $LG(\cdot, x_o) = -\delta_{x_o} I_{M\times M}$
  - $G(\cdot, x_o)\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0$
  - $\mathcal{N}_\kappa G(\cdot, x_o) \in L^1\left(\mathbb{R}^{n-1}, \dfrac{dx'}{1+|x'|^{n-1}}\right)$

  Doesn't always exist.

Example: $\Delta - 2\nabla\text{div}$ - no Poisson kernel or Green function.
Goal: Add a new concept to this list - *distinguished coefficient tensors*.

Coefficient Tensors

With each family $A \in \left[\mathbb{C}^{M\times M}\right]^{n\times n}$ associate the system L_A:

$$A = \left(A_{rs}\right)_{1\leq r,s\leq n} \rightsquigarrow L_A := A_{rs}\partial_r\partial_s$$

Given L weakly elliptic system, consider its collection of coefficient tensors

$$\mathfrak{A}_L := \{A : L_A = L\}.$$

Note that if the family $B = \left(B_{rs}\right)_{1\leq r,s\leq n}$ is antisymmetric in r, s then $L_A = L_{A+B}$. Thus

$$\#\mathfrak{A}_L \ \text{ is } \textit{infinite}.$$

Example: $L := \Delta$ in $\Omega \subseteq \mathbb{R}^2$. Then $n = 2$ and $M = 1$ and

$$\mathfrak{A}_\Delta = \left\{\begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}; \qquad \alpha \in \mathbb{C}\right\}$$

Distinguished Coefficient Tensors

Theorem (MMM - GHA Series)

Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ and let $E_{L^\top}$ be the fundamental solution of $L^\top$. Then the following are equivalent:

(a) *there exists $A \in \mathfrak{A}_L$ such that $\partial_\xi^{A^\top} E_{L^\top}(x) = 0$ for each $\xi \in \mathbb{R}^n$ and each $x \in \mathbb{R}^n \setminus \{0\}$ with $\xi \perp x = 0$.*

(b) *there exists a unique $k_L : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}^{M\times M}$ of class $\mathscr{C}^\infty$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $s \in \{1, \dots, n\}$:*

$$\int_{S^{n-1}} k_L \, d\mathscr{H}^{n-1} = I_{M\times M} \quad \text{and} \quad L(x_s k_L(x)) = 0.$$

Here, if $\xi = (\xi_r)_{1\leq r\leq n}$ and $B = (B_{rs})_{1\leq r,s\leq n}$ then $\partial_\xi^B := \xi_r B_{rs} \partial_s$.

$$\mathfrak{A}_L^{\text{dis}} := \{A \in \mathfrak{A}_L \ \text{ s.t. (a) holds.}\}$$

Basic Properties of Distinguished Coefficient Tensors

- $\mathfrak{A}_L^{\text{dis}}$ is a convex set (so if $\#\mathfrak{A}_L^{\text{dis}} \geq 2$ then $\#\mathfrak{A}_L^{\text{dis}} = \infty$)

Let L, L' be homogeneous, 2nd-order, complex constant coefficient weakly elliptic systems in $\mathbb{R}^n$. Say that $L \sim L'$ if L' is obtained from L by means of finitely many operations of the following type:

- $L \rightsquigarrow CL$ where $C \in \mathbb{C}^{M\times M}$ is invertible.
- $L \rightsquigarrow LC$ where $C \in \mathbb{C}^{M\times M}$ is invertible.
- $L \rightsquigarrow L \circ W := A_{rs}(W\nabla)_r(W\nabla)_s$ where $W \in \mathbb{R}^{n\times n}$ is invertible.

Then if $L \sim L'$ it follows that

$$\mathfrak{A}_L^{\text{dis}} \neq \varnothing \iff \mathfrak{A}_{L'}^{\text{dis}} \neq \varnothing.$$

Indeed if $C \in \mathbb{C}^{M\times M}$ and $W \in \mathbb{R}^{n\times n}$ are invertible:

- $A \in \mathfrak{A}_L^{\text{dis}} \iff AC \in \mathfrak{A}_{LC}^{\text{dis}}$ and $A \in \mathfrak{A}_L^{\text{dis}} \iff CA \in \mathfrak{A}_{CL}^{\text{dis}}$.
- $A \in \mathfrak{A}_L^{\text{dis}} \iff W^\top \circ A \circ W \in \mathfrak{A}_{L\circ W}^{\text{dis}}$.

Distinguished Coefficient Tensors

Not every weakly elliptic system posses a DCT. Indeed:

Theorem (MMM - GHA Series)

For each $n \in \mathbb{N}$ with $n \geq 2$, the $n \times n$ system $L := \Delta - 2\nabla\mathrm{div}$ in $\mathbb{R}^n$ is weakly elliptic, second-order, homogeneous, constant real coefficient, symmetric, and has the property that $\mathfrak{A}_L^{\mathrm{dis}} = \mathfrak{A}_{L^\top}^{\mathrm{dis}} = \varnothing$.

The scalar case: $L = \mathrm{div} A \nabla$ weakly elliptic. Then:

- if $n \geq 3$ then $\mathfrak{A}_L^{\mathrm{dis}} = \{\mathrm{sym}\, A\}$ where $\mathrm{sym}\, A := (A + A^\top)/2$
- if $n = 2$ then:

$$\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing \iff \mathfrak{A}_L^{\mathrm{dis}} = \{\mathrm{sym}\, A\} \iff \int_{S^1} \frac{d\mathscr{H}^1(\xi)}{L(\xi)} \neq 0$$

The case of systems:

- the complex Lamé-like system in the weak ellipticity regime:

$$L_{\mu,\lambda} := \mu\Delta + (\mu+\lambda)\nabla\mathrm{div} \quad \text{with} \quad \mu \neq 0 \quad \text{and} \quad 2\mu+\lambda \neq 0,$$

  There holds: $\mathfrak{A}^{\mathrm{dis}}_{L_{\mu,\lambda}} \neq \varnothing \iff 3\mu+\lambda \neq 0.$

- generic weakly elliptic systems: if $\mathfrak{A}^{\mathrm{dis}}_{L} \neq \varnothing$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ then $\mathfrak{A}^{\mathrm{dis}}_{L} = \{A\}$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \{A^\top\}$ for some $A \in \mathfrak{A}_L$ (hence both $\mathfrak{A}^{\mathrm{dis}}_{L}$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top}$ are singletons).
- Legendre-Hadamard elliptic systems: either $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$, or $\mathfrak{A}^{\mathrm{dis}}_{L} = \{A\}$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \{A^\top\}$ for some $A \in \mathfrak{A}_L$.

Distinguished Coefficients lead to Poisson Kernels

Theorem (MMM - GHA Series)

Let L be an $M \times M$ homogeneous constant complex coefficient second-order weakly elliptic system in $\mathbb{R}^n$ such that $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$. Fix $A \in \mathfrak{A}_L^{\text{dis}}$ and bring in $k_L : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}^{M\times M}$, associated with A as in the previous result. Then $P^L : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M\times M}$ defined by $P^L(x') := 2k_L(x',1)$ is a Poisson kernel for L in $\mathbb{R}^n_+$, i.e. it satisfies:

(a) $\exists\, C \in (0,\infty)$ *such that* $|P^L(x')| \leq \frac{C}{(1+|x'|^2)^{\frac{n}{2}}}$ *for each* $x' \in \mathbb{R}^{n-1}$.

(b) P^L *is measurable and* $\displaystyle\int_{\mathbb{R}^{n-1}} P^L(y')\,dy' = I_{M\times M}$.

(c) *If* $K^L(x',t) := P_t^L(x') = t^{1-n}P^L(x'/t)$ *for each* $x' \in \mathbb{R}^{n-1}$ *and* $t > 0$, *then* K^L *satisfies*

$$LK^L = 0_{M\times M} \quad in \quad \big[\mathcal{D}'(\mathbb{R}^n_+)\big]^{M\times M}.$$

Distinguished Coefficient Tensors

Conjectures (second order case):

- for each L weakly elliptic $\#\mathfrak{A}_L^{\rm dis} \leq 1$
- for each L Legendre Hadamard elliptic $\#\mathfrak{A}_L^{\rm dis} = 1$
- $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ if and only if $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$ (ok if $n = 2$)
- $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ if and only if L the satisfies Shapiro-Lopatinskiĭ condition
- if $\|L_1 - L_2\| \leq \varepsilon$, then $\mathfrak{A}_{L_1}^{\rm dis} \neq \varnothing$ if and only if $\mathfrak{A}_{L_2}^{\rm dis} \neq \varnothing$ (ok if $n = 2$)
- $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ if and only if Fatou holds

Conjectures (higher order case):

- $\mathfrak{A}_{\Delta^m}^{\rm dis} \neq \varnothing$ (ok if $m = 1, 2, 3, 4$)
- $\mathfrak{A}_{Lam\acute{e}^m}^{\rm dis} \neq \varnothing$ if $\lambda + 3\mu \neq 0$ (ok if $m = 1$)

Distinguished Coefficients - Higher Order Case

Fix $m \in \mathbb{N}$ and consider the operator of order $2m$ in $\mathbb{R}^n$:

$$L := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta} \partial^\beta \quad \rightsquigarrow \quad E \text{ fundamental solution}$$

Then the coefficient tensor $\{A_{\alpha\beta}\}_{|\alpha|=|\beta|=m}$ is called *distinguished* if for all $\lambda, \gamma \in \mathbb{N}_0^n$ with $|\lambda| = |\gamma| = m-1$ and all $\xi, x \in \mathbb{R}^n \setminus \{0\}$ s.t. $\xi \perp x$ there holds:

$$\sum_{|\alpha|=|\beta|=m} \sum_{\substack{\delta+\eta+e_j=\alpha \\ \theta+\omega+e_k=\gamma \\ |\delta|=|\theta| \\ \theta+\eta=\lambda}} \frac{\alpha!}{|\alpha|!} \frac{\gamma!}{|\gamma|!} \frac{|\delta|!}{\delta!} \frac{|\eta|!}{\eta!} \frac{|\theta|!}{\theta!} \frac{|\omega|!}{\omega!} \times$$

$$\times \Big[\xi_j (\partial^{\delta+\omega+\beta+e_k} E)(x) - \xi_k (\partial^{\delta+\omega+\beta+e_j} E)(x) \Big] A_{\beta\alpha}$$

$$+ \left\{ \sum_{|\alpha|=|\beta|=m} \sum_{\delta+e_j=\alpha} \frac{\alpha!}{|\alpha|!} \frac{|\delta|!}{\delta!} \xi_j (\partial^{\delta+\beta} E)(x) A_{\beta\alpha} \right\} \delta_{\lambda\gamma} = 0$$

(2) Geometry - Ahlfors Regular Domains

Here $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Definition

Call a non-empty open set $\Omega \subseteq \mathbb{R}^n$ an Ahlfors regular domain if:

(1) $\partial\Omega$ is an Ahlfors regular set, i.e., there exists $C \geq 1$ such that

$$C^{-1} R^{n-1} \leq \mathscr{H}^{n-1}\big(B(x,R) \cap \partial\Omega\big) \leq C\, R^{n-1}$$

for each $x \in \partial\Omega$ and each $R \in (0, \operatorname{diam} \partial\Omega)$,

(2) $\mathscr{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, where $\partial_*\Omega$ is the GMT boundary of Ω,

$$\partial_*\Omega := \Big\{ x \in \partial\Omega : \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap \Omega)}{r^n} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \setminus \Omega)}{r^n} > 0 \Big\}.$$

Classes of Domains

Following Hofmann, M. Mitrea & Taylor, call Ω a UR domain if:

- Ω is an Ahlfors regular domain
- $\partial\Omega$ is an uniformly rectifiable (UR) set (G. David & S. Semmes)

That is, $\exists\ \varepsilon$, $M \in (0,\infty)$ such that for each $x \in \partial\Omega$ and $0 < R < \operatorname{diam}\Omega$ one can find a Lipschitz map $\varphi : B'_R \to \mathbb{R}^n$ (where B'_R is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, and such that

$$\mathscr{H}^{n-1}\big(B(x,R) \cap \partial\Omega \cap \varphi(B'_R)\big) \geq \varepsilon R^{n-1}.$$

Call $\Omega \subset \mathbb{R}^n$ a domain without sudden big turns if Ω is an AR domain and

- when $\partial\Omega$ bounded: Ω is δ-infinitesimally flat for δ small, i.e. $\operatorname{dist}(\nu, \mathrm{VMO}(\partial\Omega)) < \delta \ll 1$,
- when $\partial\Omega$ unbounded: $\|\nu\|_{\mathrm{BMO}}$ small.

Flatness - Domains Without Sudden Big Turns

If $\Omega \subset \mathbb{R}^n$ is an AR domain, then Ω is UR follows from:

- the proximity of ν to $\mathrm{VMO}(\partial\Omega)$ when $\partial\Omega$ bounded,
- the smallness of $\|\nu\|_{\mathrm{BMO}}$ when $\partial\Omega$ unbounded.

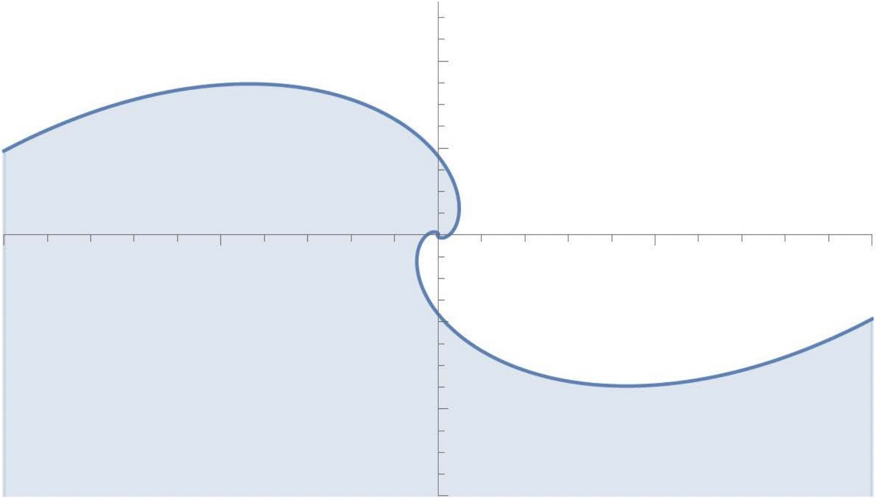
In fact, Ω becomes two-sided NTA in the sense of Jerison & Kenig and there are even topological implications. Indeed if $\|\nu\|_{\mathrm{BMO}} \ll 1$ then:

- $\partial\Omega$ is unbounded and both Ω and $\mathbb{R}^n \setminus \Omega$ are connected.

Flatness in the unbounded boundary case is typically associated with the ability to contain said set in a narrow strip - this is **not** *the case for our work, which is more inclusive:*

- the domain can have arbitrarily high peaks
- the domain can have arbitrarily deep valleys
- what is needed is a sufficiently gentle slope
- can develop gently rotating spiral points

Flatness - Domains Without Sudden Big Turns



Flatness - Domains Without Sudden Big Turns

(3) Functional Analysis - Generalized Banach Function Spaces

Nestor M. Rivière - Proceedings of Symposia in Pure Mathematics, Volume XXXV, Part 1 - AMS, Providence, Rhode Island, 1979: in connection with the following boundary value problems:
$\Omega \subseteq \mathbb{R}^n$ bounded $\mathscr{C}^1$ domain.

$$\begin{cases} \Delta^2 u = 0 \ \text{ in } \ \Omega, u\big|_{\partial\Omega}, \frac{\partial u}{\partial \nu} \ \text{ given} \\ \Delta^2 u = 0 \ \text{ in } \ \Omega, \frac{\partial u}{\partial \nu}, \frac{\partial^2 u}{\partial \nu^2} \ \text{ given} \\ \Delta^2 u = 0 \ \text{ in } \ \Omega, \frac{\partial^2 u}{\partial \nu^2}, \frac{\partial^3 u}{\partial \nu^3} \ \text{ given} \end{cases}$$

call for: *Prescribe classes of boundary data which give existence and uniqueness.*
An invitation to consider a more general Functional Analytical framework than L^p.

Generalized Banach Function Spaces

Let $(\mathscr{X}, \mu)$ be a sigma-finite measure space:

- $\mathfrak{M}$ the sigma-algebra of all μ-measurable subsets of $\mathscr{X}$
- $\mathscr{M}(\mathscr{X}, \mu)$ the vector space of all real-valued μ-measurable, finite μ-a.e. functions on $\mathscr{X}$
- $\mathbb{X}$ linear subspace of $\mathscr{M}(\mathscr{X}, \mu)$ and $\|\cdot\|_{\mathbb{X}}$ norm on $\mathbb{X}$.

Definition

$(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is called a GBFS on $(\mathscr{X}, \mu)$ provided:

(1) [Lattice Property] If $f \in \mathbb{X}$ and $g \in \mathscr{M}(\mathscr{X}, \mu)$ satisfy $|g| \leq |f|$ at μ-a.e. point in $\mathscr{X}$ then $g \in \mathbb{X}$ and $\|g\|_{\mathbb{X}} \leq \|f\|_{\mathbb{X}}$.

(2) [Fatou Property] If $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{X}$ and $f \in \mathscr{M}(\mathscr{X}, \mu)$ are such that $0 \leq f_j \nearrow f$ pointwise μ-a.e. on $\mathscr{X}$ as $j \to \infty$ and $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathbb{X}} < \infty$ then $f \in \mathbb{X}$ and $\|f\|_{\mathbb{X}} = \sup_{j\in\mathbb{N}} \|f_j\|_{\mathbb{X}}$.

(3) [Richness Property] $\exists\ \{X_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{M}$ such that $\bigcup_{j\in\mathbb{N}} X_j = \mathscr{X}$ and $\mathbf{1}_{X_j} \in \mathbb{X}$ for every $j \in \mathbb{N}$.

Generalized Banach Function Spaces

- for each $f \in \mathbb{X}$ one has $|f| \in \mathbb{X}$ and $\big\| \, |f| \, \big\|_{\mathbb{X}} = \|f\|_{\mathbb{X}}$
- pointwise multiplication by $b \in L^\infty(\mathscr{X}, \mu)$ is a bounded mapping from $\mathbb{X}$ into itself, with operator norm $\leq \|b\|_{L^\infty(\mathscr{X},\mu)}$
- any $\mathbb{X}$ GBFS on $(\mathscr{X}, \mu)$ is a Banach space.

Definition

Given $\mathbb{X}$ GBFS on $(\mathscr{X}, \mu)$, define the Köthe dual $\mathbb{X}'$ of $\mathbb{X}$ as

$$\mathbb{X}' := \left\{ g \in \mathscr{M}(\mathscr{X}, \mu) : \int_{\mathscr{X}} |fg| \, \mathrm{d}\mu < \infty \ \text{ for each } \ f \in \mathbb{X} \right\},$$

equipped with the norm
$\|g\|_{\mathbb{X}'} := \sup \left\{ \int_{\mathscr{X}} |fg| \, \mathrm{d}\mu : f \in \mathbb{X} \ \text{ and } \ \|f\|_{\mathbb{X}} \leq 1 \right\}$, for each $g \in \mathbb{X}'$.

By design, the following generalized Hölder inequality holds:

$$\int_{\mathscr{X}} |fg| \, \mathrm{d}\mu \leq \|f\|_{\mathbb{X}} \|g\|_{\mathbb{X}'} \ \text{ for each } \ f \in \mathbb{X}, \ g \in \mathbb{X}'.$$

- the Köthe dual $(\mathbb{X}', \|\cdot\|_{\mathbb{X}'})$ of a GBFS $\mathbb{X}$ on $(\mathscr{X}, \mu)$ is itself a GBFS on $(\mathscr{X}, \mu)$.

Theorem (MMM - GHA Series)

$\Sigma \subseteq \mathbb{R}^n$ UR *set and* $\sigma := \mathscr{H}^{n-1}\lfloor\Sigma$. *Assume* $\mathbb{X}$ *GBFS on* (Σ, σ) *s.t.*

$$\mathcal{M}_\Sigma : \mathbb{X} \to \mathbb{X} \quad \textit{and} \quad \mathcal{M}_\Sigma : \mathbb{X}' \to \mathbb{X}' \quad \textit{are bounded.}$$

Let k *be sufficiently smooth on* $\mathbb{R}^n \setminus \{0\}$, *odd and positive homogeneous of degree* $1-n$. *Define the operators acting on each function* $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$

$$T_\varepsilon f(x) := \int_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y) \quad \textit{for each } \varepsilon > 0 \textit{ and each } x \in \Sigma,$$

Theorem (Continued)

$$T_{\max} f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \quad \textit{for each } x \in \Sigma,$$
$$Tf(x) := \lim_{\varepsilon \to 0^+} T_\varepsilon f(x) \quad \textit{for } \sigma\textit{-a.e. } x \in \Sigma.$$

Then the operators $T_{\max}$, T *induce well-defined bounded mappings*

$$T_{\max}, T : \mathbb{X} \to \mathbb{X} \quad \textit{and} \quad T_{\max}, T : \mathbb{X}' \to \mathbb{X}'.$$

Moral: *in a Functional Analytic environment which is Harmonic Analysis friendly, the attempt to implement a singular integral approach for solving elliptic BVP's has a chance of success.*

Examples of Harmonic Analysis Friendly GBFS

- Lebesgue Spaces with $p \in (1, \infty)$
- Muckenhoupt Weighted Lebesgue Spaces with $p \in (1, \infty)$
- Variable Exponent Lebesgue Spaces (for suitable exponents)
- Lorentz Spaces
- Morrey Spaces and Block Spaces
- Muckenhoupt Weighted Morrey and Block Spaces
- Standard, Geometric and Composite Herz Spaces
- Orlicz Spaces
- Zygmund Spaces
- Spaces $L^p \exp(\log^\theta L)$ with $p \in (1, \infty)$
- Rearrangement Invariant Banach Function Spaces

(4) How it all comes together - The Dirichlet Problem

• $\Omega \subseteq \mathbb{R}^n$ open set

• let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$

• fix $\mathbb{X}$, a harmonic analysis friendly GBFS on $(\partial\Omega, \sigma)$, and $\kappa > 0$

In this context consider the BVP:

$$(D_{\mathbb{X}}) \quad \begin{cases} u \in \big[C^\infty(\Omega)\big]^M \\ Lu = 0 \ \text{ in } \ \Omega \\ \mathcal{N}_\kappa u \in \mathbb{X} \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \mathbb{X} \end{cases}$$

Goal: study well-posedness for $(D_{\mathbb{X}})$, i.e.: $\exists$, !, estimates, regularity, integral representation formulas

Set $\sigma := \mathscr{H}^{n-1}\lfloor\partial\Omega$ and consider $A \in \mathfrak{A}_L$ s.t. $L = L_A$ and E the fundamental solution of L. For $f \in \left[L^1\left(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\right)\right]^M$ define the bdry-to-domain double layer potential at each point $x \in \Omega$ as

$$\mathcal{D}_A f(x) := -\int_{\partial\Omega} \nu_s(y)\,(\partial_r E)\,(x-y) A_{sr} f(y)\,d\sigma(y)$$

and the bdry-to-bdry double layer at σ-a.e. point $x \in \partial\Omega$ by

$$K_A f(x) := -\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_s(y)\,(\partial_r E)\,(x-y) A_{sr} f(y)\,d\sigma(y)$$

Properties of the double layer potentials

As a result of the Calderón-Zygmund theory developed in GBFS if Ω is a UR domain in $\mathbb{R}^n$:

- $L(\mathcal{D}_A f) = 0$ in Ω
- $\exists\, C \in (0,\infty)$ s.t. $\|\mathcal{N}_\kappa\big(\mathcal{D}_A f\big)\|_{\mathbb{X}} \leq C\|f\|_{\mathbb{X}}$ for all $f \in \mathbb{X}$
- for every $f \in \mathbb{X}$ one has

$$(\mathcal{D}_A f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f \quad \sigma\text{-a.e. in } \partial\Omega,$$

  where I is the identity operator.

In conclusion:

- for solving $(D_{\mathbb{X}})$, it is relevant *to invert/study the spectrum of the operator* $\frac{1}{2}I + K_A$ on $\mathbb{X}$

This is an issue affected by the choice of coefficient tensor A.

The case of the Laplacian in 2D

Example: $L := \Delta$ in a domain $\Omega \subseteq \mathbb{R}^2$. Two choices of coefficient tensors in $\mathfrak{A}_\Delta$ are

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Then, for σ-a.e. $x \in \partial\Omega$:

$$K_{A_0} f(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y - x \rangle}{|x-y|^2} f(y)\, d\sigma(y),$$

i.e., the classical harmonic boundary-to-boundary double layer.
Under the natural identification $\mathbb{R}^2 \equiv \mathbb{C}$, for σ-a.e. $z \in \partial\Omega$:

$$K_{A_1} f(z) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

i.e., the boundary-to-boundary Cauchy integral operator.

The case of the Laplacian in 2D

Specialize matters to the case when $\Omega := \mathbb{R}^2_+$ and $\mathbb{X} = L^p(\partial\Omega)$ for some $p \in (1, \infty)$. In this case $\partial\Omega = \mathbb{R}$ and $\sigma = \mathcal{L}^1$ and:

- $K_{A_0} \equiv 0$ and $\frac{1}{2}I + K_{A_0} = \frac{1}{2}I$ is *trivially invertible* on $L^p(\partial\Omega)$.
- For $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$

$$K_{A_1} f(x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \mathbb{R} \\ |x-y| > \varepsilon}} \frac{f(y)}{y - x}\, dy,$$

i.e., $K_{A_1} = \frac{i}{2} H$, where H is *the classical Hilbert transform on* $\mathbb{R}$.

However $H^2 = -I$ on $L^p(\partial\Omega)$ implies $\big(K_{A_1}\big)^2 = 4^{-1} I$, and thus

$$\big(\tfrac{1}{2}I + K_{A_1}\big)\big(-\tfrac{1}{2}I + K_{A_1}\big) = 0 \;\text{ on }\; L^p(\partial\Omega),$$

precluding $\frac{1}{2}I + K_{A_1}$ from being invertible on $L^p(\partial\Omega)$.

Moral: The choice of the coefficient tensor *strongly influences* the functional analytic properties of the bdry-to-bdry potential.

Coefficient tensors $A \in \mathfrak{A}_L^{\mathrm{dis}}$ yield double layer potentials K_A which are decisively "better" in terms of recognizing flatness.

Theorem (MMM - GHA Series)

Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and suppose $A \in \mathfrak{A}_L$. TFAE:

(i) *The coefficient tensor A belongs to $\mathfrak{A}_L^{\mathrm{dis}}$.*

(ii) *If Ω is a half-space in $\mathbb{R}^n$, then the double layer $K_A \equiv 0$.*

(iii) $\exists\, k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *even, positive homogeneous of degree $-n$, and such that for any Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, the integral kernel of K_A has the form*

$$\langle \nu(y), x-y\rangle k(x-y) \quad \textit{for each } x \in \partial\Omega \textit{ and } \mathscr{H}^{n-1}\textit{-a.e. } y \in \partial_*\Omega,$$

where ν is the GMT outward unit normal to Ω.

Chord-Dot-Normal SIOs

Definition (GHA Series)

A SIO operator T is said to have a chord-dot-normal structure if

$$Tf(x) := \text{p.v.} \int_{\partial\Omega} \langle \nu(y), x - y \rangle\, k(x-y) f(y) d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega,$$

with $k \in [C^\infty(\mathbb{R}^n \setminus \{0\})]^{M\times M}$ even, positive homog. of degree $-n$.

Thus

$$A \in \mathfrak{A}_L^{\mathrm{dis}} \implies K_A \text{ has chord-dot-normal structure.}$$

This is a crucial step as the chord-dot-normal structure is critical (if and only if) in ensuring that the SIO is sensitive to the flatness of the domain.

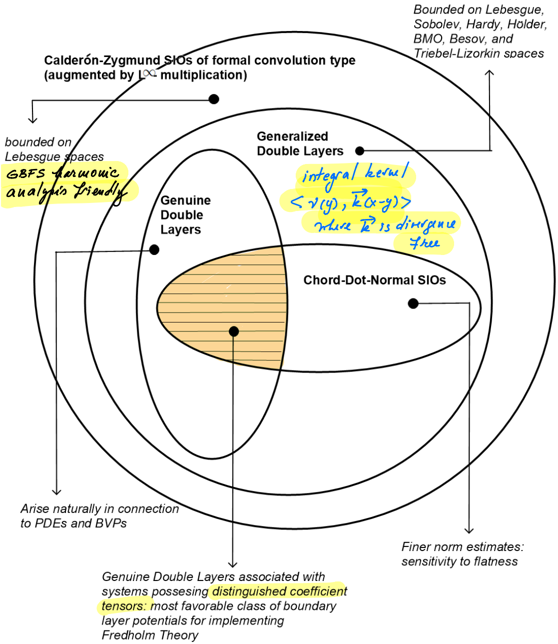
Chord-Dot-Normal SIOs

Theorem (MMM - GHA Series)

Let T be a chord-dot-normal SIO associated with $\Omega \subset \mathbb{R}^n$, a domain without sudden big turns, and $\mathbb{X}$ a harmonic analysis friendly GBFS. Then T:

- *is close to compact on $\mathbb{X}$ if $\partial\Omega$ is bounded;*
- *has small norm on $\mathbb{X}$ if $\partial\Omega$ unbounded.*

Conclusion: Both cases lead to either Fredholmness with index zero or invertibility of the operator $\frac{1}{2}I + K_A$ on $\mathbb{X}$.


Bounded on Lebesgue, Sobolev, Hardy, Hölder, BMO, Besov, and Triebel-Lizorkin spaces
Calderón-Zygmund SIOs of formal convolution type (augmented by L^∞-multiplication)
bounded on Lebesgue spaces
GBFS harmonic analysis friendly
Generalized Double Layers
integral kernel
$< \nu(y), \vec{k}(x-y) >$
where $\vec{k}$ is divergence free
Genuine Double Layers
Chord-Dot-Normal SIOs
Arise naturally in connection to PDEs and BVPs
Finer norm estimates: sensitivity to flatness
Genuine Double Layers associated with systems possesing distinguished coefficient tensors: most favorable class of boundary layer potentials for implementing Fredholm Theory

In a GMT setting:

- BVPs and SIOs on Riemannian Manifolds
- Connections with Several Complex Variables
- BVPs and SIOs for Higher Order Systems
- Scattering Theory and GMT
- Complex Analytic Methods for Elliptic PDEs in the Plane