Recent Advances in Harmonic Analysis and Partial Differential Equations

# Boundary Problems in the Plane: Answering Gelfand's Question

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## Abstract

In the 1950's, A. Bitsadze discovered that the classical Dirichlet boundary value problem for the square of the Cauchy-Riemann operator is not well posed in the unit disk. This unexpected phenomenon led I. Gelfand to attempt an explanation based on the nature of the roots of the associated characteristic equation. In this presentation, however, we indicate that the actual reason lies in an entirely different direction, and we characterize all such pathological scalar operators in the plane, as well as all pathological $2\times 2$ systems in the plane, based on powerful structural results. Moreover, we establish an equivalence between the pathological behavior of scalar operators and that of systems. This is joint work with Dorina Mitrea and Marius Mitrea.


## Bitsadze's Operator

It has been known that the $L^p$-Dirichlet BVP for real scalar operators is well posed, e.g., the laplacian $\Delta := 4\bar\partial\partial$, where $\bar\partial$ is the Cauchy-Riemann operator. The situation is dramatically different for scalar complex operators.

Bitsadze's operator $\bar\partial^2$ yields an $L^p$-Dirichlet BVP which has infinitely many linearly independent null-solutions in $\mathbb{R}^2_+$. For example,

$$u_k(z) := \frac{\operatorname{Im} z}{(z+i)^k} \quad \text{for each } z\in\mathbb{R}^2_+ \text{ and } k\in\mathbb{N} \text{ with } k\geq 2, \tag{1}$$

is such a family of null-solutions. Indeed, $u_k = 0$ on $\partial\mathbb{R}^2_+ \equiv \mathbb{R}$ and

$$\bar\partial^2 u_k(z) = 0 \quad \text{for each } z\in\mathbb{R}^2_+ \text{ and } k\in\mathbb{N} \text{ with } k\geq 2. \tag{2}$$

So, uniqueness for the $L^p$-Dirichlet BVP for $\bar\partial^2$ in $\mathbb{R}^2_+$ fails dramatically. This fact seemed at that time unexpected and almost unbelievable, and it became a subject of discussions for many mathematicians trying to explain this phenomenon.

## Ellipticity

Fix $n$, $M\in\mathbb{N}$ with $n\geq 2$. Consider a second-order, homogeneous, constant complex coefficient, $M\times M$ system in $\mathbb{R}^n$

$$L = A_{jk}\partial_j\partial_k \quad \text{with each } A_{jk}\in\mathbb{C}^{M\times M}. \tag{3}$$

Define the **symbol** of $L$ as

$$L(\xi) := -\xi_j\xi_k A_{jk}\in\mathbb{C}^{M\times M}, \quad \forall \xi = (\xi_j)_{1\leq j\leq n}\in\mathbb{R}^n.$$

We say that $L$ is **weakly elliptic** if $L(\xi)\in\mathbb{C}^{M\times M}$ is invertible for each $\xi\in\mathbb{R}^n\setminus\{0\}$, and call $L$ **Legendre-Hadamard elliptic** if $\operatorname{Re} L(\xi)\in\mathbb{R}^{M\times M}$ is strictly positive definite for each $\xi\in\mathbb{R}^n\setminus\{0\}$.

**Remark 1.** Legendre-Hadamard ellipticity implies weak ellipticity.

## Poisson Kernels

Let $L$ be a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M\times M$ system in $\mathbb{R}^n$. A **Poisson kernel** for $L$ in $\mathbb{R}^n_+$ is a matrix-valued function $P:\mathbb{R}^{n-1}\to\mathbb{C}^{M\times M}$ satisfying:

(a) There exists a constant $C\in(0,\infty)$ such that

$$|P(x)| \leq \frac{C}{(1+|x|^2)^{n/2}} \quad \text{for each } x\in\mathbb{R}^{n-1}. \tag{4}$$

(b) The function $P$ is Lebesgue measurable and

$$\int_{\mathbb{R}^{n-1}} P(x)\,dx = I_{M\times M}. \tag{5}$$

(c) If for each $x\in\mathbb{R}^{n-1}$ and $t>0$ one sets

$$K(x,t) := P_t(x) = t^{1-n}P(x/t) \tag{6}$$

then the matrix-valued function $K:\mathbb{R}^n_+\to\mathbb{C}^{M\times M}$ satisfies

$$LK = 0\cdot I_{M\times M} \quad \text{in } [\mathcal{D}'(\mathbb{R}^n_+)]^{M\times M}. \tag{7}$$

**Remark 2.** By elliptic regularity, any Poisson kernel is $\mathscr{C}^\infty$-smooth.

**Theorem 3** (Agmon-Douglis-Nirenberg; 1950's). *Every Legendre-Hadamard elliptic system has a Poisson kernel.*

## Equivalence of Systems

Let $L = A_{jk}\partial_j\partial_k$, with each $A_{jk}\in\mathbb{C}^{M\times M}$, be an $M\times M$ system in $\mathbb{R}^n$. We define the following three types of transformations:

(i) $L\mapsto PL = (PA_{jk})\partial_j\partial_k$ for any $P\in\mathbb{C}^{M\times M}$ nonsingular.

(ii) $L\mapsto LQ = (A_{jk}Q)\partial_j\partial_k$ for any $Q\in\mathbb{C}^{M\times M}$ nonsingular.

(iii) $L\mapsto L\circ W := A_{jk}(W\nabla)_j(W\nabla)_k$ for any $W\in\mathbb{R}^{n\times n}$ nonsingular.

Two systems $L_1$ and $L_2$ are said to be **equivalent** if one can be transformed into the other by means of finitely many successive applications of transformations (i)-(iii). In such a case, write $L_1\sim L_2$.

**Remark 4.** $L_1\sim L_2$ iff there exist $P,Q\in\mathbb{C}^{M\times M}$ and $W\in\mathbb{R}^{n\times n}$ all nonsingular such that $L_1 = P(L_2\circ W)Q$.

**Remark 5.** If $L_1$ and $L_2$ are scalar operators, then $L_1\sim L_2$ iff there exist $\alpha\in\mathbb{C}\setminus\{0\}$ and $W\in\mathbb{R}^{n\times n}$ nonsingular with $L_1 = \alpha(L_2\circ W)$.

**Remark 6.** Weak ellipticity is preserved under equivalence.

**Theorem 7** (<u>Structure Theorem for Scalar Operators</u>). *For any second-order, homogeneous, constant complex coefficient, weakly elliptic, scalar operator $L$ in $\mathbb{R}^2$, there is a unique $\beta\in[-1,1]\setminus\{0\}$ such that*

$$L\sim S_\beta := \partial_x^2 + i(\beta-1)\partial_x\partial_y + \beta\partial_y^2. \tag{8}$$

*Moreover, all equivalence classes are distinct, and $S_\beta$ has a Poisson kernel iff $\beta>0$.*

**Theorem 8** (<u>Structure Theorem for Real $2\times 2$ Systems</u>). *Let $L$ be a second-order, homogeneous, constant real coefficient, weakly elliptic $2\times 2$ system in $\mathbb{R}^2$. Then $L$ may be equivalently reduced to one of the following two canonical types:*

(i) *Type 1: For some $k\in(0,1]$ and $\rho\in[-k,k]\setminus\{0,k^2\}$ one has*

$$L\sim L_{k,\rho} := \begin{pmatrix}1&0\\0&\frac{\rho}{k^2}\end{pmatrix}\partial_x^2 + \begin{pmatrix}0&\frac{\rho-k^2}{k}\\\frac{\rho-1}{k}&0\end{pmatrix}\partial_x\partial_y + \begin{pmatrix}\rho&0\\0&1\end{pmatrix}\partial_y^2. \tag{9}$$

(ii) *Type 2: For some $b\in\{0,1\}$ and $\tau\in(0,1]$, one has*

$$L\sim L^{b,\tau} := \begin{pmatrix}1&0\\0&1\end{pmatrix}\partial_x^2 + 2\begin{pmatrix}0&0\\b&0\end{pmatrix}\partial_x\partial_y + \begin{pmatrix}1&0\\0&\tau\end{pmatrix}\partial_y^2. \tag{10}$$

*Moreover, there is no overlap between type 1 and type 2 and all equivalence classes are distinct within both type 1 and type 2.*

*Finally, $L_{k,\rho}$ has a Poisson kernel iff $\rho\neq -k$, while $L^{b,\tau}$ has a Poisson kernel for all $b,\tau$ in the specified sets above.*

Recall the matrix representations of any complex number $z=x+iy$:

$$z^\flat := \begin{pmatrix}x\\y\end{pmatrix} \quad\text{and}\quad z^\sharp := \begin{pmatrix}x&-y\\y&x\end{pmatrix}. \tag{11}$$

For any second-order, homogeneous, complex scalar operator in $\mathbb{R}^n$

$$L = a_{jk}\partial_j\partial_k \quad \text{with each } a_{jk}\in\mathbb{C}, \tag{12}$$

define the second-order, homogeneous, real $2\times 2$ system in $\mathbb{R}^n$

$$L^\sharp := a^\sharp_{jk}\partial_j\partial_k. \tag{13}$$

**Remark 9.** If $L_1$ and $L_2$ are two complex scalar operators, then $L_1\sim L_2$ if and only $L_1^\sharp\sim L_2^\sharp$ (this is surprisingly delicate).

## The $L^p$-Dirichlet Problem

Fix an integrability exponent $p\in(1,\infty)$. Let $L$ be a second-order, homogeneous, constant complex coefficient, $M\times M$ system in $\mathbb{R}^n$. The **$L^p$-Dirichlet Boundary Value Problem** for $L$ in $\mathbb{R}^n_+$ is formulated for a fixed aperture parameter $\kappa\in(0,\infty)$ as

$$\begin{cases} u\in[\mathscr{C}^\infty(\mathbb{R}^n_+)]^M,\\ Lu=0 \ \text{ in } \mathbb{R}^n_+,\\ \mathcal{N}_\kappa u\in L^p(\mathbb{R}^{n-1}),\\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f\in[L^p(\mathbb{R}^{n-1})]^M. \end{cases} \tag{14}$$

The **space of admissible boundary data** for the $L^p$-Dirichlet boundary value problem for the operator $L$ in the upper-half space is

$$\mathscr{AD}^p_L := \Big\{ f\in[L^p(\mathbb{R}^{n-1})]^M : \text{ the } L^p \text{ Dirichlet BVP for } L \text{ in } \mathbb{R}^n_+ \text{ with boundary datum } f \text{ has a solution}\Big\}. \tag{15}$$

The **space of null-solutions** for the $L^p$-Dirichlet boundary value problem for $L$ in $\mathbb{R}^n_+$ is defined as

$$\mathscr{NS}^p_L := \Big\{ u\in[\mathscr{C}^\infty(\mathbb{R}^n_+)]^M : u \text{ solves the } L^p \text{ Dirichlet BVP for } L \text{ in } \mathbb{R}^n_+ \text{ with boundary datum } 0\Big\}. \tag{16}$$

**Theorem 10** (<u>All-or-Nothing Theorem for Scalar Operators</u>). *Let $L$ be a second-order, homogeneous, constant complex coefficient, weakly elliptic, scalar operator in $\mathbb{R}^2$. Then $\mathscr{AD}^p_L$ is closed and*

$$\text{either } \ \mathscr{AD}^p_L = L^p(\mathbb{R}) \ \text{ and } \ \mathscr{NS}^p_L = \{0\},$$
$$\text{or } \ \mathscr{AD}^p_L = \Big\{ f\in L^p(\mathbb{R}) : Hf = \pm if\Big\} \ \text{ and } \dim \mathscr{NS}^p_L = \infty,$$

*where $H$ is the Hilbert transform on the real line.*

**Theorem 11** (<u>All-or-Nothing Theorem for Real $2\times 2$ Systems</u>). *Let $L$ be a second-order, homogeneous, constant real coefficient, weakly elliptic, $2\times 2$ system in $\mathbb{R}^2$. Then $\mathscr{AD}^p_L$ is closed and*

$$\text{either } \ \mathscr{AD}^p_L = [L^p(\mathbb{R})]^2 \ \text{ and } \ \mathscr{NS}^p_L = \{0\},$$
$$\text{or } \ \mathscr{AD}^p_L = \Big\{ f\in [L^p(\mathbb{R})]^2 : \mathbb{H}f = \pm if\Big\} \ \text{ and } \dim \mathscr{NS}^p_L = \infty,$$

*where $\mathbb{H} := -iHi^\sharp = \begin{pmatrix}0&iH\\-iH&0\end{pmatrix}$ is the matrix Hilbert transform.*

**Theorem 12.** *Let $L$ be second-order, homogeneous, constant complex coefficient, scalar operator in $\mathbb{R}^2$. Then one has*

$$\mathscr{AD}^p_L = \Big\{ g+ih : (g,h)\in\mathscr{AD}^p_{L^\sharp} \text{ with } g \text{ and } h \text{ real-valued}\Big\}$$
$$\text{and } \ \mathscr{AD}^p_{L^\sharp} = \Big\{ g^\flat + ih^\flat : g,h\in\mathscr{AD}^p_L\Big\}.$$

**Theorem 13.** *Let $L$ be second-order, homogeneous, constant complex coefficient, scalar operator in $\mathbb{R}^n$. Then one has*

$$\mathscr{NS}^p_L = \Big\{ g+ih : (g,h)\in\mathscr{NS}^p_{L^\sharp} \text{ with } g \text{ and } h \text{ real-valued}\Big\}$$
$$\text{and } \ \mathscr{NS}^p_{L^\sharp} = \Big\{ g^\flat + ih^\flat : g,h\in\mathscr{NS}^p_L\Big\}.$$

**Theorem 14.** *Let $L$ be a second-order, homogeneous, constant complex coefficient, weakly elliptic, scalar operator in $\mathbb{R}^2$. Then the following statements hold:*

(i) *The $L^p$-Dirichlet BVP for $L$ in $\mathbb{R}^2_+$ is well posed if and only if the $L^p$-Dirichlet BVP for $L^\sharp$ in $\mathbb{R}^2_+$ is well posed.*

(ii) *$L$ has a Poisson kernel iff $L^\sharp$ has a Poisson kernel (in $\mathbb{R}^2_+$).*

(iii) *For each $\eta\in[0,1)$, one has $L\not\sim\bar\partial^2+\eta\partial^2$ iff $L^\sharp\not\sim(\bar\partial^2+\eta\partial^2)^\sharp$.*

(iv) $\displaystyle\int_{S^1}\frac{1}{L(\xi)}\,d\mathcal{H}^1(\xi)\neq 0$ *iff* $\det\Big[\int_{S^1}[L^\sharp(\xi)]^{-1}\,d\mathcal{H}^1(\xi)\Big]\neq 0$.

**Theorem 15** (<u>Main Theorem for Scalar Operators</u>). *Let $L$ be a second-order, homogeneous, constant complex coefficient, weakly elliptic, scalar operator in $\mathbb{R}^2$. Then the following are equivalent:*

(i) *For some (or any) $p\in(1,\infty)$, the $L^p$-Dirichlet boundary value problem for $L$ in $\mathbb{R}^2_+$ is well posed.*

(ii) *$L$ has a Poisson kernel in $\mathbb{R}^2_+$.*

(iii) *$L\not\sim\bar\partial^2+\eta\partial^2$ for any $\eta\in[0,1)$.*

(iv) $\displaystyle\int_{S^1}\frac{1}{L(\xi)}\,d\mathcal{H}^1(\xi)\neq 0$.

**Theorem 16** (<u>Main Theorem for Real $2\times 2$ Systems</u>). *Let $L$ be a second-order, homogeneous, constant real coefficient, $2\times 2$ system in $\mathbb{R}^2$. Then the following conditions are equivalent:*

(i') *For some (or any) $p\in(1,\infty)$, the $L^p$-Dirichlet boundary value problem for $L$ in $\mathbb{R}^2_+$ is well posed.*

(ii') *$L$ has a Poisson kernel in $\mathbb{R}^2_+$.*

(iii') *$L\not\sim(\bar\partial^2+\eta\partial^2)^\sharp$ for any $\eta\in[0,1)$.*

(iv') $\displaystyle\det\Big[\int_{S^1}[L(\xi)]^{-1}\,d\mathcal{H}^1(\xi)\Big]\neq 0$.

**Remark 17.** Collectively, Theorems 15-16 answer Gelfand's question.

**Remark 18.** As a corollary of Theorems 15-16, in the class of complex scalar operators and real $2\times 2$ systems in $\mathbb{R}^2$, the property of possessing a Poisson kernel is preserved under equivalence.

**Remark 19.** Theorems 15-16 also imply that, in the class of complex scalar operators and real $2\times 2$ systems in $\mathbb{R}^2$, the well-posedness of the $L^p$-Dirichlet problem is stable under small perturbations of $L$.

**Remark 20.** Similar results to Theorems 15-16 also hold for the Homogeneous and Inhomogeneous Regularity Problems in $\mathbb{R}^2_+$.

## Example: The Lamé System

**Example 21.** Recall the Lamé system of elasticity in $\mathbb{R}^2$,

$$\mathcal{L}_{\lambda,\mu} = \mu\Delta\cdot I_{2\times 2} + (\lambda+\mu)\nabla\operatorname{div}, \quad \lambda,\mu\in\mathbb{R}. \tag{17}$$

Then $\mathcal{L}_{\lambda,\mu}$ is weakly elliptic iff $\mu\neq 0$ and $\lambda+2\mu\neq 0$. Staying in this regime, define $t := \frac{\mu}{\lambda+2\mu}\in\mathbb{R}\setminus\{0\}$. Then:

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\sim L_{1,-1} \text{ (type 1 without Poisson kernel)} & \text{if } t=-1\\ \mathcal{L}_{\lambda,\mu}\sim L^{0,1} \text{ (type 2 with Poisson kernel)} & \text{if } t=1\\ \mathcal{L}_{\lambda,\mu}\sim L_{1,1/t} \text{ (type 1 with Poisson kernel)} & \text{if } |t|>1\\ \mathcal{L}_{\lambda,\mu}\sim L_{1,t} \text{ (type 1 with Poisson kernel)} & \text{if } |t|<1. \end{cases} \tag{18}$$

Therefore,

$$\begin{aligned} \lambda+3\mu\neq 0 &\iff \mathcal{L}_{\lambda,\mu} \text{ has a Poisson kernel in } \mathbb{R}^2_+\\ &\iff \text{for some (or any) } p\in(1,\infty),\\ &\qquad \text{the } L^p\text{-Dirichlet BVP for } \mathcal{L}_{\lambda,\mu} \text{ in } \mathbb{R}^2_+ \text{ is well posed.} \end{aligned}$$

This is also seen from Theorem 16 and the fact that

$$\int_{S^1}[\mathcal{L}_{\lambda,\mu}(\xi)]^{-1}d\mathcal{H}^1(\xi) = \frac{\pi(\lambda+3\mu)}{\mu(\lambda+2\mu)}\cdot I_{2\times 2}. \tag{19}$$

## References


[1] Loo-Keng, H., Wei, L., Ci-Quian, W., *Second-Order Systems of Partial Differential Equations in the Plane*, Research Notes in Mathematics (1985).

[2] Mitrea, D., Mitrea, I., Mitrea, M., *Geometric Harmonic Analysis - Volumes I-V*, Developments in Mathematics (Springer Nature, Cham, 2022).

[3] Verchota, G. C., Vogel, A.L., *Nonsymmetric Systems on Nonsmooth Planar Domains*, Transactions of the American Mathematical Society (1997).