THE DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS WITH A SINGULAR DRIFT TERM

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Abstract: We establish solvability of the L^p Dirichlet problem, for some $p < \infty$, for elliptic equations of the form

$$Lu := -\text{div}\,(A\nabla u) + \mathbf{b}\cdot\nabla u =: L_0 u + \mathbf{b}\cdot\nabla u = 0\,,$$

in a 1-sided chord arc domain $\Omega \subset \mathbb{R}^{n+1}$ (i.e., a uniform domain with an Ahlfors-David regular boundary), provided that L^p solvability holds (typically for a different p) for the homogeneous second order equation $L_0 u = 0$, and (roughly speaking) that the drift term satisfies $\text{dist}(X,\partial\Omega)|\mathbf{b}(X)| \lesssim 1$, as well as the Carleson measure condition

$$\iint_{\Omega\cap B(x,r)} |\mathbf{b}(Y)|^2 \text{dist}(Y,\partial\Omega)\, dY \lesssim r^n, \quad x \in \partial\Omega\,, 0 < r < \text{diam}(\partial\Omega).$$

In previous work with J. L. Lewis, we had treated the analogous problem for parabolic equations in the half-space (and hence, via a pull-back mechanism, in certain parabolic Lipschitz graph domains), and had presented a claimed, simpler proof in the elliptic case, based on an erroneous proof of doubling of elliptic measure for operators with drift terms. In fact, it appears that doubling remains an open problem (except in the small constant case), so the putative elliptic argument in that work was incorrect.

The proof presented in the present work provides a purely elliptic argument, still simpler than that in the parabolic case, which does not require doubling (again, except in the small constant case). Moreover, the present argument allows the treatment of much more general domains, whose boundaries need not be given locally as graphs.

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