$\kappa$-toposes

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WAECO 2024
Rasiowa-Sikorski: Boolean-algebras

Proof:

\[ b \geq b_1 \geq b_2 \geq \ldots, \text{ s.t.} \]

- if \( b_n \cap \neg x_{n+1} \neq 0 \) \( \Rightarrow \) that is \( b_{n+1} \)
- if \( b_n \cap \neg x_{n+1} = 0 \) \( \Rightarrow b_n \preceq x_{n+1} \Rightarrow b_n \cap x_{n+1,i} \neq 0 \) \( \Rightarrow \) that is \( b_{n+1} \).

f.i.p.

Kristóf Kanalas

\( \kappa \)-toposes

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Theorem

$L$ is a finitely complete lattice with $0$, \((x_n, i \rightarrow x_n)_i\)\(n<\omega\) given \(\cap\)-stable unions.

Then for every \(b \neq 0\) there's a filter \(F \ni b\), \(F \nsubseteq 0\) which preserves each given union.
proof:

\[ \uparrow b \text{ is a filter. goal: } b \geq b_1 \geq \ldots \]

s.t. \( \bigcup (\uparrow b \subseteq \uparrow b_1 \subseteq \ldots) \) is good.

\( T_b \) is "tasks concerning \( b \)" = set of 
\( (x_{n,i} \rightarrow x_n)_i \) with \( x_n \geq b \)
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\[ T_b \text{ is "tasks concerning } b" = \text{set of } (x_{n,i} \rightarrow x_n)_{i} \text{ with } x_n \geq b \]

List \( T_b \) to the zeroth column of an \( \omega \times \omega \)-table, solve task \((0,0)\), i.e. intersect this cover with \( b \), choose one leg \( b_1 \).

Fill \( T_{b_1} \) in the first column, solve task \((0,1)\), etc.
Any such choice $\bigcup(\uparrow b \subseteq \uparrow b_1 \subseteq \ldots)$ preserves the given unions, but may not preserve 0.

Need:

not every $\omega$-branch is becoming eventually 0. More is true: if we cut it down we get a cover on $b$. 

\[ b \rightarrow \cdots \rightarrow \cdots \]

\[ \cdots \rightarrow b \rightarrow \cdots \]
Rasiowa-Sikorski: $\kappa$-complete lattices with 0

Infinitary version: $(\aleph_0 \leq \kappa = cf(\kappa)).$
Rasiowa-Sikorski: $\kappa$-complete lattices with 0

Infinitary version: $\aleph_0 \leq \kappa = \text{cf}(\kappa)$.

**Definition**

Let $L$ be a $\kappa$-complete lattice with 0. Take a set $E$ of unions which happen to exist. These unions are compatible with $< \kappa$ limits if:

- They are $\cap$-stable. (The resulting set of unions is $E^\cap$.)
- Given a continuous tree glued together from $E^\cap$-families, s.t. every branch is $< \kappa$, the branches cover the root.
Rasiowa-Sikorski: $\kappa$-complete lattices with 0

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**Theorem**

Let $L$ be a $\kappa$-complete lattice with 0, $E$ a $\leq \kappa$ big set of unions compatible with $< \kappa$ limits, $L \ni b \neq 0$.

Then there is a filter $F$ s.t. $F \ni b$, $F \neq 0$, closed under $< \kappa$ meets, preserves the given unions.

proof: $\bigcup_{i<\kappa} \uparrow b_i$ is $\kappa$-closed. Use the canonical well-ordering of $\kappa \times \kappa$. □
Rasiowa-Sikorski: $\kappa$-complete categories

One can fatten up the proof:

$L \xrightarrow{\sim} \kappa$-complete category $C$

- $\prec \kappa$ meets $\kappa$-limits
- $\kappa$-closed filter $\xrightarrow{\sim} \kappa$-lex functor $C \to \text{Set}$
- unions $\xrightarrow{\sim} \text{exremal epimorphic families}$
- $0 \xrightarrow{\sim} \text{strict initial object}$
- $\uparrow b \xrightarrow{\sim} C(b, -)$
Rasiowa-Sikorski: $\kappa$-complete categories

One can fatten up the proof:

\[
\begin{array}{ccc}
L & \xrightarrow{\text{< $\kappa$ meets}} & \text{unions} & \xrightarrow{\text{0}} & \uparrow b \rightarrow \\
\kappa\text{-closed filter} & \xrightarrow{\text{< $\kappa$ limits}} & \kappa\text{-lex functor $C \rightarrow Set$} & \xrightarrow{\text{strict initial object}} & C(b, -)
\end{array}
\]

**Theorem**

$C$ is a $\kappa$-complete category, $\forall x, y : |C(x, y)| \leq \kappa$. Let $E$ be a $\leq \kappa$ big set of extremal epimorphic families, s.t. $E$ is compatible with $< \kappa$ limits. Then for every $i : u \nRightarrow x$ there's an $M : C \rightarrow Set$ $\kappa$-lex functor which sends the $E$-families to jointly surjective ones, and keeps $M(i)$ proper.

This gives a completeness theorem for $L^g_{\lambda\kappa}$. I’ll try to argue why.
Categories in logic

**Definition**

$L$ is a $\kappa$-ary signature. $(\aleph_0 \leq \kappa = cf(\kappa) \leq \lambda = cf(\lambda))$

$(\lambda, \kappa)$-geometric formula: atomics, $\bigwedge \bullet < \bullet \kappa$, $\bigvee \bullet < \bullet \lambda$, $\exists \bullet < \bullet \kappa$.

$L^g_{\lambda, \kappa}$ has formulas: $\forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$ $\varphi$, $\psi$ are $(\lambda, \kappa)$-geom. $(|\bar{x}| < \kappa)$
Categories in logic

**Definition**

$L$ is a $\kappa$-ary signature. ($\aleph_0 \leq \kappa = cf(\kappa) \leq \lambda = cf(\lambda)$)

$(\lambda, \kappa)$-geometric formula: atomics, $\land \bullet < \bullet < \kappa$, $\lor \bullet < \bullet < \lambda$, $\exists \bullet < \bullet < \kappa$.

$L_{\lambda, \kappa}^g$ has formulas: $\forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$, $\varphi$, $\psi$ are $(\lambda, \kappa)$-geom. ($|\bar{x}| < \kappa$)

$M$ is an $L$-structure $\rightsquigarrow \text{Def}_{\lambda, \kappa}(M)$: category of $(\lambda, \kappa)$-geometric definable sets and definable functions.
Observation: $Def_{\lambda, \kappa}(M)$ is closed under some universal constructions, e.g.:

- $\prec \kappa$ products: $(\langle \varphi_i(x_i) \rangle^M)_{i < \gamma < \kappa}$ their product is $[\bigwedge_i \varphi_i(x_i)]^M$ (renamed variables).
- Image factorization:

\[
\begin{array}{ccc}
\varphi(x)^M & \xrightarrow{\mu(x, y)^M} & \psi(y)^M \\
\mu(x, y')^M & \Rightarrow & \exists x \mu(x, y')^M \\
\end{array}
\]

\[\exists x \mu(x, y') \land y' = y]^M\]

- Also: equalizers, $\prec \lambda$ unions.
Observation: $\text{Def}_{\lambda, \kappa}(M)$ is closed under some universal constructions, e.g.:

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& \Rightarrow & [\exists x \mu(x, y') \land y' = y]^M
\end{array}
$$

- Also: equalizers, $\lambda$ unions.

Idea: $\text{Def}_{\lambda, \kappa}(M)$ is the $\text{ev}_M$-image of some abstract ”category of formulas”, these constructions live there, $\text{ev}_M$ preserves them.
Fact: given $T \subseteq L^g_{\lambda,\kappa}$ we replace it with: $(C, E)$, s.t.
- $C$ is a $\kappa$-complete category
- $E$ is a collection of extremal epimorphic families, compatible with $< \kappa$ limits.

$\text{Mod}(T) = \{ C \rightarrow \textbf{Set} \ \kappa\text{-lex } E\text{-preserving} \}$
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proof idea:
objects: formulas, arrows: $T$-provably functional formulas.
preserving limits/unions/images $\Rightarrow$ preserving the formula constructors.
Fact: given $T \subseteq L_{\lambda,\kappa}^g$ we replace it with: $(C, E)$, s.t.

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$$\text{Mod}(T) = \{ C \to \textbf{Set} \ \kappa\text{-lex } E\text{-preserving } \}$$

Proof idea:
Objects: formulas, arrows: $T$-provably functional formulas.
Preserving limits/unions/images = preserving the formula constructors.

Conversely: every such $(C, E)$ gives a theory in $L_{\lambda,\kappa}^g$. Moreover $E$ can be arbitrary. (Then the completeness theorem takes a different form.)

$\hookrightarrow \kappa$-toposes.
Topology on a category: collection $E$ of families, s.t:
- closed under pullbacks
- closed under $ht=2$ trees
- contains the isos

Sheaves: $C^\text{op} \rightarrow \textbf{Set}$ functors, s.t. sections glue together and are uniquely determined by restrictions on covers.
Definition

$(C, E)$ $\kappa$-site: $C$ is $\kappa$-lex, $E$ is a set of families.

$\langle E \rangle_\kappa$ (generated $\kappa$-topology): closure of $E$ under pullbacks and continuous trees (with $< \kappa$ long branches).

Theorem

$(C, E)$ is a $\kappa$-site, $\forall x, y : |C(x, y)| \leq \kappa$, $|E| \leq \kappa$. Then there is

$\text{Sh}(C, \langle E \rangle_\kappa) \to \text{Set}^I \kappa$-lex, continuous, conservative.

$L^g_{\infty, \kappa}$-theories $\models \kappa$-toposes, and this is a completeness theorem.