1. Let $y = f(x)$, $x \in \mathbb{R}$, be sufficiently smooth and

\[
\Delta_+ f(x) = \frac{f(x + p) - f(x)}{p}, \quad \Delta_- f(x) = \frac{f(x) - f(x - q)}{q},
\]

where $0 < p, q \ll 1$, $p \neq q$.

(a) Show that in general $\Delta_+ \Delta_- f(x) = \frac{q}{p} (\Delta_- \Delta_+ f(x))$.

(b) Derive a consistent finite difference formula approximating $f''(x)$ utilizing the steps $p, q$, $p \neq q$, for $x \in \mathbb{R}$. What is the order of convergence of your formula above?

2. Consider the numerical solution of the IVP

\begin{equation}
y' = f(t, y), \quad t \geq t_0, \quad y(t_0) = y_0, \tag{1.1}
\end{equation}

where $f$ satisfies a Lipschitz condition in the second variable.

(a) Show that the following explicit finite difference method is convergent for solving (1.1).

\[
y_{n+1} - y_n - hf(t_n, y_n), \quad n = 0, 1, 2, \ldots
\]

(b) Show that the following implicit Runge-Kutta method for solving (1.1) is $A$-stable.

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3. Consider the following IBVP:

\begin{align*}
&u_t = u_{xx} + \kappa u_x + f(x), \quad -1 < x < 1, \quad t > t_0; \tag{3.1} \\
u(-1, t) = u(1, t) = 0, \quad t \geq t_0; \tag{3.2} \\
u(x, 0) = \phi(x), \quad -1 < x < 1, \tag{3.3}
\end{align*}

where $\kappa \in \mathbb{R}$ is a constant and $f, \phi$ are sufficiently smooth on $(-1, 1)$.

(a) Derive a semi-discretized scheme for solving (3.1)-(3.3) based on a central finite difference approximation to the spatial derive on uniform grids. Show that your scheme is consistent.

(b) Derive a fully discretized scheme based on your semidiscretized scheme and a backward Euler method. Show the consistency.