NAME: ______________________

APPLIED MATHEMATICS I
2024 QUALIFYING EXAM

- YOU HAVE 90 MINUTES
- SHOW YOUR WORKS WITH ENOUGH EXPLANATIONS
1. Answer the following questions.
(a) State the Lax-Milgram lemma.
(b) Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with polygonal boundary. Let \( \mathbf{b} \) be a vector field \( \mathbf{b} = (b_1, b_2) \) with functions \( b_i \in C^1(\Omega) \cap C(\Omega), 1 \leq i \leq 2 \) such that \( \text{div} \mathbf{b} = 0 \). Consider the variational problem finding \( u \in H^1_0(\Omega) \) such that
\[
B(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1_0(\Omega)
\]
for \( f \in L^2(\Omega) \). Show that this variational equation has a unique solution.

2. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with polygonal boundary. Consider the boundary value problem
\[
-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]
with given function \( f \in L^2(\Omega) \). Let \( V = H^1_0(\Omega) \).
(a) Derive (with intermediate steps) a variational equation of (0.2) with a bilinear form on \( V \) and a linear functional on \( V \).
(b) Suppose that \( V_h \subset V \) is a finite element space with continuous piecewise linear polynomials for a triangulation \( T_h \) of \( \Omega \). State the variational equation finding numerical solution \( u_h \in V_h \).
(c) Assuming that \( u \in H^2(\Omega) \), prove that \( \| \text{grad}(u - u_h) \|_{L^2(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \) where \( h > 0 \) is the maximum diameter of the triangles in \( T_h \) and \( C > 0 \) is a constant independent of \( h \).

3. Let
\[
\Sigma = H(\text{div}, \Omega) := \{ \tau \in L^2(\Omega; \mathbb{R}^d) : \text{div} \tau \in L^2(\Omega) \}, \quad W = L^2(\Omega),
\]
and introduce an additional variable \( \sigma = -\text{grad} u \).
(a) Rewrite (0.2a) as a system of two equations with two variables \( \sigma \) and \( u \).
(b) Derive a variational equation of the system with a bilinear form on \( \Sigma \times W \) and a linear functional on \( \Sigma \times W \).

4. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with polygonal boundary. Assume that \( V_h \subset H^1_0(\Omega) \) is a finite element space with continuous piecewise linear polynomials. Consider a semidiscrete problem finding \( u_h : (0, T) \to V_h \) satisfying
\[
\int_{\Omega} (\partial_t u_h(t)) v \, dx + \int_{\Omega} \nabla u_h(t) \cdot \nabla v \, dx = \int_{\Omega} f(t) v \, dx
\]
for all \( 0 < t < T \) and for all \( v \in V_h \).
(a) Assuming that numerical initial data \( u^0_h \in V_h \) is given, state the Crank–Nicolson scheme with time step size \( \Delta t > 0 \) for (0.3).
(b) If we say \( u^n_h \) be the numerical solution at \( n \)-th time step of the Crank–Nicolson scheme, prove that
\[
\| u^n_h \|_{L^2(\Omega)} \leq \| u^0_h \|_{L^2(\Omega)} + C\Delta t \sum_{i=0}^{n-1} \| f(i\Delta t) \|_{L^2(\Omega)}
\]
with \( C > 0 \) independent of \( n \). (The Gronwall inequality is not assumed to be known)