Are the following propositions true or false? Give a short proof or provide a counterexample. Answer four out of the following six!

(i) Let $\mathcal{M}$ be the $\sigma$-algebra of all Lebesgue measurable subsets of $\mathbb{R}$. Then $\text{card}(\mathbb{R}) = \text{card}(\mathcal{M})$.

(ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function and $f(x) = g(x)$ almost everywhere. Then $g$ is also Lebesgue measurable.

(iii) Suppose $f(x)$ is Riemann integrable on $[0, 1]$ and $f(q) = q$ for every $q \in \mathbb{Q} \cap [0, 1]$. Then its Lebesgue integral is given by $\int_{[0,1]} f \, dm = 1/2$.

(iv) Let $M, N \subseteq \mathbb{R}^2$ be two sets of (two-dimensional) Lebesgue measure zero. Then $M + N = \{m + n | m \in M, n \in N\}$ also has Lebesgue measure zero.

(v) If $f \in L^p(\mathbb{R})$, where $p \in [1, \infty)$, then
\[ \lim_{n \to \infty} \int_n^{n+1} f(x) \, dx = 0 \, . \]

(vi) If $f \in L^p(\mathbb{R})$, where $p \in (1, \infty)$, then
\[ \lim_{n \to \infty} \int_n^{\infty} f(x) \, dx = 0 \, . \]
Answer four out of the following five problems! If you use any of the theorems proven in class, carefully state it and make sure to show that its assumptions are satisfied.

1.) (a) Compute the following:
\[
\lim_{n \to \infty} \int_{[0, \infty)} \frac{x}{1 + x^n} dm(x) .
\]

(b) Let \( f \) be non-negative and \( f \in L^1(0,1) \). Let \( 0 < \alpha < 1 \). Define
\[
g(x) = \int_{0}^{1} \frac{f(y)}{|x-y|^\alpha} dy \quad \forall x \in [0,1] .
\]
Prove \( g \in L^1(0,1) \) and estimate \( \|g\|_1 \) in terms of \( \|f\|_1 \).

2.) (a) Prove that the equation
\[
f(x) + \frac{1}{2} \int_{0}^{x} e^{-\sin(y)} f(y)dy = e^x
\]
has a unique solution in \( L^2(0,1) \) (no need to determine it).

(b) Consider the linear operator \( T : L^1(0,1) \to L^1(0,1) \) given by
\[
(Tf)(x) = x^2 \int_{0}^{1} f(t)dt .
\]
Prove that \((I - T)\) is boundedly invertible and find an integral operator \( K \) such that \((I - T)^{-1} = I + K\). Is \( K \) compact?

3.) (a) Let \( f \in L^p(\mathbb{R}) \), where \( p \in (1, \infty) \). For any \( n \in \mathbb{N} \), define \( f_n(x) := f(x-n) \). Prove that \((f_n)_{n \in \mathbb{N}}\) converges weakly to zero in \( L^p(\mathbb{R}) \).

(b) Prove that if \( f \in L^2(0,1) \) and
\[
\int_{0}^{1} fg dm = 0
\]
for every \( g \in C([0,1]) \), then \( f = 0 \) a.e.

4.) Let \( X \) be a Banach space and let \( S : X \supseteq \mathcal{D}(S) \to X \) be a closed operator. Recall that the graph norm \( \| \cdot \|_S \) of \( S \) is given by \( \|f\|_S := \|f\| + \|Sf\| \) for all \( f \in \mathcal{D}(S) \).

(a) Let \( \Gamma > 0 \) be an arbitrary positive number. Prove that \( \| \cdot \|_{S,\Gamma} \) given by \( \|f\|_{S,\Gamma} = \Gamma \|f\| + \|Sf\| \) for all \( f \in \mathcal{D}(S) \) is equivalent to \( \| \cdot \|_S \).

(b) Let \( T : X \supseteq \mathcal{D}(T) \to X \) be a linear operator such that \( \mathcal{D}(S) \subseteq \mathcal{D}(T) \) and there exist \( a > 0 \) and \( 0 < b < 1 \) such that
\[
\|Tf\| \leq a\|f\| + b\|Sf\|
\]
for all \( f \in \mathcal{D}(S) \). Prove that the operator \((S + T)\) given by

\[
(S + T) : X \supseteq \mathcal{D}(S + T) = \mathcal{D}(S) \to X, \quad f \mapsto Sf + Tf
\]

is closed.

5.) The goal of this problem is to show that if \( f \in L^1(0,1) \), then

\[
\lim_{n \to \infty} \int_0^1 f(x) \sin(nx) \, dx = 0.
\]

(\(\ast\))

(a) Show that

\[
\lim_{n \to \infty} \int_a^b \sin(nx) \, dx = 0
\]

for any \( 0 \leq a < b \leq 1 \) and thus conclude that (\(\ast\)) holds for every \( f \) that is a step function.

(b) Using that the step functions are dense in \( L^1(0,1) \) (you don’t need to prove this) conclude (\(\ast\)) for every \( f \in L^1(0,1) \).