Instructions: Complete seven of the following problems. Start a new page for each problem.

1. Let G be a finite group and let p be a prime number.
   (a) How is the center of G defined? $Z(G) =$
   (b) How is the centralizer of an element $x \in G$ defined? $C_G(x) =$
   (c) State and prove the class equation for G.
   (d) Show that if G is a nontrivial p-group, then $Z(G)$ is nontrivial.

2. Let G be a finite group whose order is divisible by the prime number p.
   (a) What is a Sylow p-subgroup of G?
   (b) State the three Sylow theorems for G.
   (c) What else can we say about the number of Sylow p-subgroups of G if we assume that G is a simple nonabelian group?
   (d) Show that there is no simple group of order 80.

3. Let R be a commutative ring with $1 \neq 0$.
   (a) Complete the definition: An ideal I of R is a maximal ideal if ...
   (b) Show that R has at least one maximal ideal.
   (c) Show that an ideal I of R is a maximal ideal if and only if R/I is a field.

4. Let F be a field.
   (a) Briefly explain why the polynomial ring $F[x]$ is a PID.
   (b) Show that if $f(x) \in F[x]$ is a nonzero polynomial of degree n, then $f(x)$ has at most n roots in F.
   (c) Show that if F is finite, then the multiplicative group $F^\times = F - \{0\}$ is cyclic.

5. Let F be a field and let $f(x) \in F[x]$ be a nonconstant polynomial.
   (a) Define what it means for $f(x)$ to be solvable by radicals over F. As part of your answer, also define what we mean by a radical field extension of F.
   (b) Define the Galois group of $f(x)$ and state Galois' theorem.
   (c) Show that the polynomial $f(x) = x^5 - 6x^2 + 3$ is not solvable by radicals over $\mathbb{Q}$. You may take use (without proof) that $f(x)$ has exactly three distinct real roots and one pair of complex conjugate roots.

6. Let $A \in M_5(\mathbb{C})$ and suppose that the Smith normal form of $tI - A \in M_5(\mathbb{C}[t])$ is

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & t+1 & \\ & & & & (t+1)^2(t-2)^2 \end{pmatrix}.$$

(a) Find the minimal polynomial of A.

(b) Find the characteristic polynomial of A.

(c) Find the rational normal form of A.

(d) Find the Jordan normal form of A.

(e) View $V = \mathbb{C}^5$ as a $\mathbb{C}[t]$-module so that $tv = Av$ for all $v \in V$. Use the fundamental theorem for finitely generated modules over a PID to explain why V is not a cyclic $\mathbb{C}[t]$-module.

7. Let R be a commutative ring.

(a) Show that if $\pi : M \to N$ and $\sigma : N \to M$ are R-module homomorphisms such that $\pi \circ \sigma = \mathrm{id}_N$, then $M = \ker \pi \oplus \operatorname{im} \sigma$.

(b) Show that if $0 \to L \to M \to N \to 0$ is a short exact sequence of R-module homomorphisms and N is a free R-module, then $M \cong L \oplus N$.

8. Let R be a ring and let L be a simple left R-module.

(a) Show that $\mathrm{End}_R(L)$ is a division ring.

(b) Show that if R is a $\mathbb{C}$-algebra and $\dim_{\mathbb{C}}(L) < \infty$, then $\mathrm{End}_R(L) \cong \mathbb{C}$.

(c) Suppose R is a semisimple $\mathbb{C}$-algebra and $\dim_{\mathbb{C}}(R) < \infty$. Use part (b) and the Wedderburn–Artin theorem to show that $R \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$.