1. Complete the following definitions (carefully)

(a) A subbasis for a topology on the set $X$ is a collection $B$ of subsets of $X$ such that ...

(b) Let $X$ and $Y$ be topological spaces and $q : X \to Y$ be continuous. Then $q$ is a quotient map if ...

(c) Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. A basis for the product topology on $\prod_{\alpha \in J} X_\alpha$ is given by ...

(d) A topological space $X$ is locally connected if ...

(e) A topological space $X$ is compact if ...

(f) Let $X$ be a topological space. We say $X$ is first countable if ...

(g) Let $X$ be a topological space. Two functions $f, f' : [0, 1] \to X$ are path homotopic if ...

(h) Let $X$ and $Y$ be a topological spaces. $X$ is a covering space of $Y$ if ...

(i) Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is nullhomotopic if ...
2. Complete the following definitions (carefully)

   (a) Let $X$ be a topological space. A singular $k$-simplex is …

   (b) Let $X$ be a topological space. The group $C_k(X)$ of singular $k$-chains is …

   (c) Let $X$ be a topological space. The boundary map $\partial_k : C_k(X) \to C_{k-1}(X)$ is given by …

   (d) Let $(X, A)$ be a pair of topological spaces. The group $C_k(X, A)$ of relative $k$-chains is …

   (e) Let $(X, A)$ be a pair of topological spaces. The boundary map $\partial_k : H_k(X, A) \to H_{k-1}(X)$ in the long exact sequence of the pair $(X, A)$ is given by …

   (f) Let $X$ be a CW complex. The group $C^\text{CW}_k(X)$ of cellular $k$-chains is …

   (g) Let $X$ be a CW complex. The boundary map $d_k : C^\text{CW}_k(X) \to C^\text{CW}_{k-1}(X)$ is given by …

   (h) Let $X$ be a topological space and let $G$ be an abelian group. The group $C^k(X; G)$ of singular $k$-cochains with coefficients in $G$ is …

   (i) Let $X$ be a topological space and let $R$ be a commutative ring. The cup product

   \[ \cup : C^i(X, R) \times C^j(X; R) \to C^{i+j}(X; R) \]

   of singular cochains with coefficients in $R$ is given by …
3. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

(a) Let $X$ and $Y$ be compact spaces. If $N$ is an open subset of $X \times Y$ and contains $\{x_0\} \times Y$, then there exists a neighborhood $W$ of $x_0$ with $W \times Y \subseteq N$.

(b) Every metrizable space is normal.

(c) If $X$ is a compact Hausdorff space, then $X$ is a Baire space.

(d) Let $\pi : E \to B$ be a covering map with $\pi (e_0) = b_0$ and $\gamma : I \to B$ a path beginning at $b_0$. Then $\gamma$ lifts to a path $\tilde{\gamma} : I \to B$ beginning at $e_0$. (Nb: you do not need to prove the uniqueness of the lift.)
4. Complete **TWO** of the following problems.

   (a) Let $X$ be a topological space.
      
      i. Let $\{A_n\}_{n \in \omega}$ be a collection of connected subspaces of $X$ and suppose that for each $n \in \omega$, $A_n \cap A_{n+1} \neq \emptyset$. Show that $\bigcup A_n$ is connected.
      
      ii. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of connected subspaces of $X$. Let $A$ be a connected subspace of $X$ with $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$. Prove that $A \cup \bigcup A_\alpha$ is connected.
   
   (b) Let $X$ be a topological space and $A \subseteq X$. Show that if $x \in X$ and there is a sequence in $A$ which converges to $x$, then $x \in A$. Is the converse true? (prove it or give a counterexample)
   
   (c) Prove that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^2$.
   
   (d) Show that a countable product of metric spaces is metric. Explain (briefly) why this does not hold for uncountable products
5. Complete **ALL** of the following problems.

(a) Prove that there is no retraction from the closed unit disc onto the unit circle.

(b) Let $X$ be the $n$-fold dunce cap, i.e. the quotient space formed from the closed unit disc by identifying each point $z$ of the unit circle with the points $r(x), r^2(x), \ldots, r^{n-1}(x)$ where $r : S^1 \rightarrow S^1$ is given by rotation by $2\pi/n$. Find $\pi_1(X)$ (provide an informal justification).

(c) Show that space $\theta = S^1 \cup \{(x,0) : -1 \leq x \leq 1\}$ and the space $X = S^1 \vee S^1$ are homotopy equivalent but not homeomorphic. (provide an informal justification).
6. Complete **exactly TWO** of the following problems.

   (a) State and prove the Euler-Poincaré principle for $\Delta$-complexes.

   (b) State the Poincaré index theorem and use it to prove that every continuous (tangent) vector field on $S^2$ has at least one singular point.

   (c) State the excision theorem and use it to calculate the groups $H_k(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\})$. 
7. Complete exactly **TWO** of the following problems.

(a) Prove that the reduced homology groups of a star-shaped region in $\mathbb{R}^n$ are trivial.

(b) Use the method of your choice to calculate the homology groups $H_k(S^2 \times S^2)$.

(c) Use a Mayer-Vietoris sequence to calculate the homology groups $H_k(\mathbb{R}^3 \setminus S^1)$, where we identify $S^1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.  

8. Complete **TWO** of the following problems.

(a) Let $X$ be a topological space such that

$$H_k(X) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \text{ or } k = 3; \\
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1; \\
0 & \text{otherwise.}
\end{cases}$$

Use the universal coefficient theorem for homology to calculate the groups $H_k(X; G)$, where $G$ is an abelian group.

(b) Write (without proof) the cellular chain complex of $\mathbb{R}P^3$ with respect to its standard CW complex structure and then calculate the homology groups $H_{\text{CW}}^k(\mathbb{R}P^3) \cong H_k(\mathbb{R}P^3)$.

(c) Use the cellular chain complex of $\mathbb{R}P^3$ from part (b) to calculate the cohomology groups $H_{\text{CW}}^k(\mathbb{R}P^3; G) \cong H^k(\mathbb{R}P^3; G)$, where $G$ is an abelian group.