Topology Qualifying Exam

Spring 2022

1. Complete the following definitions (*carefully*)

   (a) A *subbasis for a topology* on the set X is a collection $\mathcal{B}$ of subsets of X such that …

   (b) Let X and Y be topological spaces and $q : X \to Y$ be continuous. Then q is a *quotient map* if …

   (c) Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. A *basis* for the *product topology* on $\prod_{\alpha \in J} X_\alpha$ is given by …

   (d) A topological space X is *locally connected* if …

   (e) A topological space X is *compact* if …

   (f) Let X be a topological space. We say X is *first countable* if …

   (g) Let X be a topological space. Two functions $f, f' : [0, 1] \to X$ are *path homotopic* if …

   (h) Let X and Y be a topological spaces. X is a *covering space of* Y if …

   (i) Let X and Y be topological spaces. A map $f : X \to Y$ is *nulhomotopic* if …

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2. Complete the following definitions (*carefully*)

   (a) Let X be a topological space. A *singular k-simplex* is …

   (b) Let X be a topological space. The group $C_k(X)$ of *singular k-chains* is …

   (c) Let X be a topological space. The *boundary map* $\partial_k : C_k(X) \to C_{k-1}(X)$ is given by …

   (d) Let (X, A) be a pair of topological spaces. The group $C_k(X, A)$ of *relative k-chains* is …

   (e) Let (X, A) be a pair of topological spaces. The *boundary map* $\partial_k : H_k(X, A) \to H_{k-1}(X)$ in the long exact sequence of the pair (X, A) is given by …

   (f) Let X be a CW complex. The group $C_k^{\mathrm{CW}}(X)$ of *cellular k-chains* is …

   (g) Let X be a CW complex. The *boundary map* $d_k : C_k^{\mathrm{CW}}(X) \to C_{k-1}^{\mathrm{CW}}(X)$ is given by …

   (h) Let X be a topological space and let G be an abelian group. The group $C^k(X; G)$ of *singular k-cochains* with coefficients in G is …

   (i) Let X be a topological space and let R be a commutative ring. The *cup product*

   $$\cup : C^i(X, R) \times C^j(X; R) \to C^{i+j}(X; R)$$

   of singular cochains with coefficients in R is given by …

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3. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) Let X and Y be compact spaces. If N is an open subset of $X \times Y$ and contains $\{x_0\} \times Y$, then there exists a neighborhood W of x_0 with $W \times Y \subseteq N$.

   (b) Every metrizable space is normal.

   (c) If X is a compact Hausdorff space , then X is a Baire space.

   (d) Let $\pi : E \to B$ be a covering map with $\pi(e_0) = b_0$ and $\gamma : I \to B$ a path beginning at b_0. Then γ lifts to a path $\widetilde{\gamma} : I \to B$ beginning at e_0. (*Nb:* you do not need to prove the uniqueness of the lift.)

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4. Complete **TWO** of the following problems.

   (a) Let X be a topological space.

      i. Let $\{A_n\}_{n\in\omega}$ be a collection of connected subspaces of X and suppose that for each $n \in \omega$, $A_n \cap A_{n+1} \neq \varnothing$. Show that $\bigcup A_n$ is connected.

      ii. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of connected subspaces of X. Let A be a connected subspace of X with $A \cap A_\alpha \neq \varnothing$ for each $\alpha \in \Lambda$. Prove that $A \cup \bigcup A_\alpha$ is connected.

   (b) Let X be a topological space and $A \subseteq X$. Show that if $x \in X$ and there is a sequence in A which converges to x, then $x \in \overline{A}$. Is the converse true? (prove it or give a counterexample)

   (c) Prove that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^2$.

   (d) Show that a countable product of metric spaces is metric. Explain (briefly) why this does not hold for uncountable products

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5. Complete **ALL** of the following problems.

   (a) Prove that there is no retraction from the closed unit disc onto the unit circle.

   (b) Let X be the n-fold dunce cap, i.e. the quotient space formed from the closed unit disc by identifying each point z of the unit circle with the points $r(x)$, $r^2(x)$, $\ldots r^{n-1}(x)$ where $r : S^1 \to S^1$ is given by rotation by $2\pi/n$. Find $\pi_1(X)$ (provide an informal justification).

   (c) Show that space $\theta = S^1 \cup \{(x, 0) : -1 \leq x \leq 1\}$ and the space $X = S^1 \vee S^1$ are homotopy equivalent but not homeomorphic. (provide an informal justification).

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6. Complete **exactly TWO** of the following problems.

   (a) State and prove the Euler-Poincaré principle for Δ-complexes.

   (b) State the Poincaré index theorem and use it to prove that every continuous (tangent) vector field on S^2 has at least one singular point.

   (c) State the excision theorem and use it to calculate the groups $H_k(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\})$.

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7. Complete exactly **TWO** of the following problems.

   (a) Prove that the reduced homology groups of a star-shaped region in $\mathbb{R}^n$ are trivial.

   (b) Use the method of your choice to calculate the homology groups $H_k(S^2 \times S^2)$.

   (c) Use a Mayer-Vietoris sequence to calculate the homology groups $H_k(\mathbb{R}^3 \setminus S^1)$, where we identify $S^1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

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8. Complete **TWO** of the following problems.

   (a) Let X be a topological space such that

   $$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = 3; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1; \\ 0 & \text{otherwise.} \end{cases}$$

   Use the universal coefficient theorem for homology to calculate the groups $H_k(X; G)$, where G is an abelian group.

   (b) Write (without proof) the cellular chain complex of $\mathbb{R}P^3$ with respect to its standard CW complex structure and then calculate the homology groups $H_k^{\mathrm{CW}}(\mathbb{R}P^3) \cong H_k(\mathbb{R}P^3)$.

   (c) Use the cellular chain complex of $\mathbb{R}P^3$ from part (b) to calculate the cohomology groups $H^k_{\mathrm{CW}}(\mathbb{R}P^3; G) \cong H^k(\mathbb{R}P^3; G)$, where G is an abelian group.