Qualifying Exam in Real Analysis

Spring 2022, June 2nd 2-5pm, Sid Richardson 210

Name:

The exam consists of two parts. You are asked to choose 4 problems out of the total 5 problems from Part I, and to choose 3 problems out of the total 4 problems of part II. Each problem worths 20 pts.

1. Part I

(1) (a) State the Lebesgue Dominated Convergence Theorem.

(b) Using the Lebesgue Dominated Convergence Theorem, prove

$$\lim_{n\to\infty} \int_0^\infty \frac{\ln(x+n)}{n} e^{-x} \cos nx dx = 0.$$

Show all your calculations.

(2) Let $(X, \mathcal{S}, \mu)$ be a measure space and suppose $f \in L^1(\mu)$. Prove that

$$\{x \in X \mid f(x) \neq 0\}$$

is the countable union of sets each with finite μ-measure.

(3) (a) State Fubini's Theorem.

(b) For $E = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < \infty, 0 < y < 1\}$, use Fubini's Theorem to compute

$$\int_E ye^{-xy} \sin x d(\lambda \times \lambda)$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Do not forget to show that

$$\int_E \left| ye^{-xy} \sin x \right| d(\lambda \times \lambda) < \infty.$$

(c) Show that

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy dx = \frac{1}{2} \text{ and } \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx dy = -\frac{1}{2}.$$

Why does this not contradict Fubini's Theorem? Prove it by showing all calculations.

(4) Prove that if A and B are disjoint subsets of $\mathbb{R}$ and B is Lebesgue measurable, then

$$|A \cup B| = |A| + |B|$$

where $|\cdot|$ denotes the outer measure.

(5) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function.

(a) For $k \in \mathbb{N}$, let

$$G_k = \{a \in \mathbb{R} \mid \exists \delta > 0 \text{ such that } |f(b) - f(c)| < \frac{1}{k} \ \forall b, c \in (a-\delta, a+\delta)\}.$$

Prove that G_k is an open subset of $\mathbb{R}$ for each $k \in \mathbb{N}$.

(b) Prove that the set of points at which f is continuous equals $\cap_{k=1}^\infty G_k$.

(c) Conclude that the set of points at which f is continuous is a Borel set.

2. Part II

(1) (a) Prove that the Banach space $L^2([0,1])$ is separable.
(b) Prove that the Banach space $L^\infty([0,1])$ is not separable

(2) (a) Prove that the closed unit ball of $\ell^2(\mathbb{N})$ is not compact.
(b) Prove that the subset

$$\{(a_k)_k \in \ell^2(\mathbb{N}); \sum_{k\in\mathbb{N}} k^2|a_k|^2 \leq 1\}$$

is compact.

(3) Suppose V is a Banach space with norm $\|\cdot\|$ and $\phi : V \mapsto \mathbb{C}$ is a linear functional. Define another norm $\|\cdot\|_\phi$ on V by

$$\|f\|_\phi = \|f\| + |\phi(f)|.$$

Prove that V is complete with respect to the norm $\|\cdot\|_\phi$ if and only if ϕ is continuous on V with respect to the original norm $\|\cdot\|$.

(4) Let

$$c_0 = \{(a_k) \in \ell_\infty; \lim_{k\mapsto\infty} a_k = 0\}.$$

Give c_0 the norm that inherits as a subspace of ℓ_∞. Prove that the dual space of c_0 can be identified with ℓ_1.