2022 Qualifying Exam, Applied II part.

Do 5 of 7.

1. a) Assume A is diagonalizable. Give a diagonalized form for A. Show using this form that A^{-1} has eigenvalues that are reciprocals of those of A and has the same eigenvectors as A.

b) Use the singular value decomposition (SVD) of A to show that A^{-1} has singular values that are reciprocals of A's.

c) Can you show that A^2 has eigenvalues that are squares of A's eigenvalues? (Show it is true or show that it is not.)

d) Same as c) for singular values of A^2 (can you show singular values of A^2 are squares of singular values of A; either show or show they are not). What about singular values of $A^T A$, can you show they related to singular values of A?

2. Consider this 1-D, elliptic boundary value problem:

$-u'' = \delta(x - \frac{1}{4})$

$u(0) = 2$

$u(1) = -1$

a) Find the finite element solution with $h = \frac{1}{2}$. (Use linear basis functions.)

b) Solve exactly (analytically instead of numerically).

3. a) Show that left and right eigenvectors corresponding to different eigenvalues are orthogonal.

b) Show for A symmetric, that $\|A\|$ equals the max eigenvalue in absolute value. (You can use standard facts such as that eigenvectors can be chosen to be orthogonal for a symmetric matrix).

c) Give the spectral decomposition formula for symmetric matrices with distinct eigenvalues.

d) If one term is deleted from the spectral decomposition in part c), show that the norm does not increase.

e) (Extra Credit) Is d) true for nonsymmetric matrices?

4. a) Consider the simple wave equation $u_t = -au_x$, $u(x, 0) = q(x)$. Solve with substitution $\tau = t$, $\xi = x - at$.

b) Can you explain why it is a "wave" equation?

c) Looking at the differential equation $u_t = -au_x$, can you explain why a wave moves (what about the partial with respect to t being related to the partial with respect to x causes this to happen)?

5. a) Use Taylor Series to derive a finite difference formula for $f''(a)$ in terms of $f(a)$, $f(a+2h)$ and $f(a-h)$. Give the leading error term.

b) Use the formula in part a) to approximately solve for the value of u at $x = 1/3$ in this differential equation ($h = \frac{1}{3}$):

$-u'' = 3x$

$u(0) = -1$

$u(1) = 2$

6. Consider finding eigenvalues with a Krylov subspace method.

a) Explain when there will be fast convergence for a particular eigenvalue and when there will be slow convergence.

b) Do a convergence analysis to show why part a) is true.

7. a) A is diagonalizable and has eigenvalues 1, 10, 20. At what eventual rates will power method and inverse power method converge?

b) What is the best shift for shifted power method if you are finding the eigenvector corresponding to the eigenvalue at 20, and at what rate will it converge?

c) For matrix A shown below (QR factorization is also shown), take one step of QR iteration. Does it make progress in the one iteration? Explain.

```
A =

    1.0000    2.0000
    0.0100         0

>> [q,r] = qr(A)

q =

   -1.0000   -0.0100
   -0.0100    1.0000


r =

   -1.0000   -1.9999
         0   -0.0200
```