NAME : ______________________

APPLIED MATHEMATICS I
2022 QUALIFYING EXAM

- YOU HAVE 120 MINUTES
- SHOW YOUR WORKS WITH ENOUGH EXPLANATIONS

1. State the Lax-Milgram lemma.

2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with Lipschitz boundary and $\Gamma_D \subset \partial\Omega$ be a nonempty open subset of $\partial\Omega$. For functions in

$$V_D = \{v \in H^1(\Omega) \, : \, v|_{\Gamma_D} = 0\}$$

state the Poincare inequality.

3. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain with polygonal/polyhedral boundary and there are two disjoint open subsets Γ_D, Γ_N of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. Consider the boundary value problem

$$-\Delta u = f \qquad \text{in } \Omega, \tag{0.1a}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{0.1b}$$

$$\frac{\partial u}{\partial \nu} = u_N \qquad \text{on } \Gamma_N. \tag{0.1c}$$

with given functions $f \in L^2(\Omega)$, $u_N \in L^2(\Gamma_N)$. Introducing a new variable $\sigma = \operatorname{grad} u$ and function spaces

$$\Sigma = L^2(\Omega; \mathbb{R}^d), \qquad V_D = \{v \in H^1(\Omega) \,:\, v|_{\Gamma_D} = 0\}$$

derive a variational equation of (0.1) with a bilinear form on $\Sigma \times V_D$ and a linear functional on $\Sigma \times V_D$.

4. For (0.1) the solution u of (0.1) satisfies

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} u_N v \, dS \qquad \forall v \in V_D. \tag{0.2}$$

Assume that $V_h \subset V_D$ is the space of piecewise linear continuous finite elements for a triangulation of Ω such that the maximum diameter of simplices is $h > 0$. Let $u_h \in V_h$ be the solution of

$$\int_\Omega \operatorname{grad} u_h \cdot \operatorname{grad} v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} u_N v \, dS \qquad \forall v \in V_h. \tag{0.3}$$

Assuming $u \in H^2(\Omega)$, prove that $\| \operatorname{grad}(u - u_h)\|_{L^2(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}$ with a constant $C > 0$ independent of h.

5. Let u be the solution of (0.1) and u_h be the solution of (0.3). Assume that $u \in H^2(\Omega)$ and for any $g \in L^2(\Omega)$ the solution ϕ of

$$-\Delta\phi = g \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega,$$

satisfies $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|g\|_{L^2(\Omega)}$ with $C_\Omega > 0$ depending only on Ω. Based on the conclusion of Problem 4 and these assumptions, prove that $\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{H^2(\Omega)}$ with $C > 0$ independent of h.

6. For a triangle $T \subset \mathbb{R}^2$ define $\mathrm{RT}(T)$, a subspace of vector-valued polynomials on T, by

$$\mathrm{RT}(T) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ y \end{pmatrix} \ : \ a, b, c \in \mathbb{R} \right\}.$$

Here x, y are the coordinates of $\mathbb{R}^2$. Let F_i, $i = 0, 1, 2$ be the edges of T, and n_i be the outward unit normal vectors on F_i, $i = 0, 1, 2$. Define the degrees of freedom of $\mathrm{RT}(T)$ by

$$\int_{F_i} \tau \cdot n_i \, dS, \qquad i = 0, 1, 2, \quad \tau \in \mathrm{RT}(T).$$

(1) Prove that an element in $\mathrm{RT}(T)$ is uniquely determined by the above degrees of freedom.

(2) Suppose that ϕ is an $\mathbb{R}^2$-valued C^1-function on T and define $\Pi_{\mathrm{RT}}\phi \in \mathrm{RT}(T)$ by

$$\int_{F_i} \Pi_{\mathrm{RT}}\phi \cdot n_i \, dS = \int_{F_i} \phi \cdot n_i \, dS, \qquad i = 0, 1, 2.$$

Prove that $\displaystyle\int_T (\operatorname{div} \Pi_{\mathrm{RT}}\phi - \operatorname{div} \phi) \, dx = 0$.

7. Let V_h be a space of piecewise continuous finite element space on Ω satisfying the Poincare inequality. Suppose that $u_h(t) \in C^1([0,T]; V_h)$ be a solution of semidiscrete heat equation satisfying

$$\int_\Omega \partial_t u_h(t) v \, dx + \int_\Omega \operatorname{grad} u_h(t) \cdot \operatorname{grad} v \, dx = \int_\Omega f(t) v \, dx \qquad \forall v \in V_h, \forall t \geq 0.$$

Show that

$$\|u_h(T)\|^2_{L^2(\Omega)} + \int_0^T \|\operatorname{grad} u_h(s)\|^2_{L^2(\Omega)} \, ds \leq \|u_h(0)\|^2_{L^2(\Omega)} + C \int_0^T \|f(s)\|^2_{L^2(\Omega)} \, ds$$

with $C > 0$ independent of T.