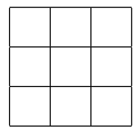
# Algebra Qualifying Exam

**Problem 1: (Section 5.3)** The following figure shows nine squares of side length 1 arranged into one square of side length 3.



If $n$ colors are available, find the number of distinct colorings of this configuration, if each of the nine squares must be colored in one single color. Two colorings will be different only if one cannot be obtained from the other by rotation and/or reflection.

**Problem 2: (Sections 5.4/5.5)** Let $G$ be a simple group of order 660.
**(a)** Find $n_{11}$, the number of distinct Sylow 11-subgroups of $G$. (Justify your answer!)
**(b)** How many elements of order 11 are there in $G$. (Justify your answer!)

**Problem 3: (Sections 4.5/4.6)** Let $A$ be the matrix

$$\begin{bmatrix} 14 & 40 & 14 \\ 4 & 20 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

over the integers.
**(a)** Find the Smith normal form of $A$. Find also matrices $P$ and $Q$ such that $QAP^{-1}$ is the Smith normal form of $A$. (Show your work!)
**(b)** Identify the abelian group given by generators $x$, $y$, $z$ and relations

$$14x + 40y + 14z = 0, \ 4x + 20y + 4z = 0, \ 2x + 4y + 2z = 0.$$

**(c)** Find the number of distinct solutions $(x, y, z)$ with $x, y, z \in \mathbb{Z}_{60}$ for the following linear system of equations. (Justify your answer!)

$$\begin{aligned} 14x + 40y + 14z &= 0, \\ 4x + 20y + \ \ 4z &= 0, \\ 2x + \ \ 4y + \ \ 2z &= 0. \end{aligned}$$

**Problem 4: (Sections 3.1/3.2/6.3)** Let $\alpha = \sqrt{2 + \sqrt{2}}$.
**(a)** Find the minimal polynomial $f$ of $\alpha$ over $\mathbb{Q}$. (Please verify irreducibility!)
**(b)** What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?
**(c)** Show that $\mathbb{Q}(\alpha)$ is the splitting field of $f$ over $\mathbb{Q}$.
**(d)** Show that $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_4$.

**Problem 5: (Sections 2.9/6.6)**
**(a)** Show that the polynomial $f(X) = X^5 - 30X^4 + 180$ is irreducible over $\mathbb{Z}$.
**(b)** Use a substitution to argue that also $g(Y) = 180Y^5 - 30Y + 1$ is irreducible over $\mathbb{Z}$.
**(c)** Find the Galois group of $f$. (Justify your answer!)

**Problem 6: (Sections 2.6/2.7/7.2/7.7)** Consider the ring $R = \mathbb{Z}[\sqrt{-13}]$.
**(a)** What are the units of $R$? (Justify your answer!)
**(b)** Show that every prime integer $0 < p < 13$ is irreducible in $R$.
**(c)** Is 7 a prime element of $R$? (Justify your answer!)
**(d)** Is $R$ a PID? Is $R$ a Dedekind domain? (Justify your answers!)