# Topology Qualifying Exam

Spring 2021

1. Complete the following definitions (*carefully*)

   (a) A *basis for a topology* on the set $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that ...

   (b) The point $x$ is a *limit point* of the set $A$ in a topological space $X$ provided that ...

   (c) Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is *continuous* if ...

   (d) Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. A *basis* for the *product topology* on $\prod_{\alpha \in J} X_\alpha$ is given by ...

   (e) Let $X$ be topological space. The *component* of $x$ in $X$ is ...

   (f) Let $X$ be a topological space. We say $X$ is *Hausdorff* if ...

   (g) Let $X$ and $Y$ be topological spaces. Two functions $f, f' : X \to Y$ are *homotopic* if ...

   (h) Let $X$ and $Y$ be a topological spaces and $f : X \to Y$ a continuous map. A set $U \subseteq Y$ is *evenly covered* by $f$ if ...

   (i) Let $X$ be a topological space and $A$ a subset of $X$. Then $A$ is a *deformation retract* of $X$ if ...

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2. Complete the following definitions (*carefully*)

   (a) Let $X$ be a topological space. The *(singular) homology group* $H_k(X)$ is ...

   (b) Let $f : X \to Y$ be a continuous map. Then $f_* : H_i(X) \to H_i(Y)$ is given by ...

   (c) Let $(X, A)$ be a pair of topological space. The *relative homology group* $H_k(X, A)$ is ...

   (d) Let $(X, A)$ be a pair of topological space. Then $\partial : H_k(X, A) \to H_{k-1}(X)$ is given by ...

   (e) A pair $(X, A)$ of topological spaces is called a *good pair* if ...

   (f) Let $X$ be a CW complex. The *cellular homology group* $H_i^{CW}(X)$ is ...

   (g) Let $f : S^n \to S^n$ be a continuous map. Then the *degree* of $f$ is ...

   (h) Let $X$ be a topological space and let $G$ be an abelian group. The *(singular) cohomology group* $H^i(X; G)$ is ...

   (i) Let $f : X \to Y$ be a continuous map. Then $f^* : H^i(Y; G) \to H^i(Y; G)$ is given by ...

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3. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) A nonempty subset $A \subseteq \mathbb{R}^n$ is compact if and only if $A$ is closed and bounded (in the standard metric on $\mathbb{R}^n$).

   (b) The product of finitely many compact space is compact. *NB: If you use any lemmas in this argument, you must also prove them*

   (c) If $X$ is a compact Hausdorff space , then $X$ is a Baire space.

   (d) Let $\pi : E \to B$ be a covering map with $\pi(e_0) = b_0$ and $\gamma : I \to B$ a path beginning at $b_0$. Then $\gamma$ lifts to a path $\widetilde{\gamma} : I \to B$ beginning at $e_0$. (*Nb:* you do not need to prove the uniqueness of the lift.)

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4. Complete **TWO** of the following problems.

   (a) Let $A, B \subseteq X$, a topological space.

      i. Show that if $A$ is connected and $A \subseteq B \subseteq \overline{A}$, then $B$ is also connected.

      ii. Prove or give a counterexample for each of the following equations: (1) $\overline{A} \cup \overline{B} = \overline{A \cup B}$ and (2) $\overline{A} \cap \overline{B} = \overline{A \cap B}$.

      iii. Show that if $A$ and $B$ are connected, then $A \times B$ is connected.

   (b) Suppose that $q : X \to Y$ is a quotient map. Prove that if $p^{-1}(y)$ is connected for each $y$ and $Y$ is connected, then so is $X$.

   (c) Show that a compact Hausdorff space is normal.

   (d) Show that a countable product of separable spaces is separable.

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5. Complete **ALL** of the following problems.

   (a) Prove that the fundamental group of the circle is isomorphic to $\mathbb{Z}$.

   (b) Let $X$ be the complement of the $z$-axis in $\mathbb{R}^3$. Find $\pi_1(X)$ (provide an informal justification).

   (c) Let $Y$ be the space obtained by removing three distinct points from $\mathbb{R}^2$. Find $\pi_1(Y)$ (provide an informal justification).

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6. Complete **exactly TWO** of the following problems.

   (a) Let $f, g : X \to Y$ be two homotopic continuous maps and let $f_*, g_* : H_k(X) \to H_k(Y)$ be the induced homomorphisms in homology. Sketch a proof that $f_* = g_*$.

   (b) State the snake lemma and explain how it is used to construct the long exact sequence in homology of a pair of topological spaces $(X, A)$.

   (c) Let $X$ be a CW complex with no two cells in adjacent dimensions. Prove that $H_k^{CW}(X)$ is a free abelian group with a basis in one-to-one correspondence with the $k$-cells of $X$.

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7. Complete exactly **TWO** of the following problems.

   (a) Use the long exact sequence of the good pair $(D^n, S^{n-1})$ to prove that

   $$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \text{ or } k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

   *Remark:* You may use without proof that the quotient $D^n/S^{n-1}$ is homeomorphic to $S^n$.

   (b) Calculate the local homology groups $H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ and then use the result to prove that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.

   (c) Prove that the antipodal map $f : S^n \to S^n$, $x \mapsto -x$, has degree $(-1)^{n+1}$.

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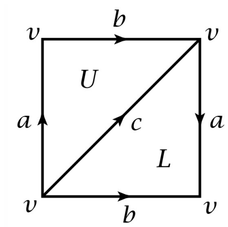
8. Complete exactly **TWO** of the following problems.

   (a) Let $K^2$ be the Klein bottle equipped with a $\Delta$-complex structure as shown below:

   Use the $\Delta$-complex structure to show that

   $$H_k(K^2) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

   (b) Let $K^2$ be the Klein bottle as above. Use the universal coefficient theorem to compute the cohomology groups $H^k(K^2; G)$, where $G$ is an abelian group.

   *Remark:* You may use without proof the following facts from homological algebra:

   - $\mathrm{Hom}(\_, G)$ and $\mathrm{Ext}^1(\_, G)$ are additive functors
   - $\mathrm{Hom}(\mathbb{Z}, G) \cong G$
   - $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G) \cong T_n(G) := \{g \in G \mid ng = 0\}$
   - $\mathrm{Ext}^1(\mathbb{Z}, G) = 0$
   - $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

   (c) Use a Mayer–Vietoris sequence to calculate the homology groups of a disk in the plane with two circular holes.