Qualifying Exam: Real Analysis

Instructions: Choose 5 problems to solve and turn in your solutions via e-mail by **May 25, 2021**. This exam is open book, open notes. You are to work *individually* on this exam–absolutely no consulting with others.

1 [10 points] Let $f : [0, \infty) \to \mathbb{R}$ be Lebesgue integrable.

(a) Show that if f is uniformly continuous, then $\lim_{x \to \infty} f(x) = 0$.

(b) Is it sufficient for f to be continous? Give a proof or a counterexample.

2 [10 points] Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a nonnegative μ-integrable function such that for each $n \in \mathbb{N}$,

$$\int f(x)^n d\mu = \int f(x) d\mu.$$

(a) Show that the set $G = \{x \in X : f(x) > 1\}$ has measure zero.

(b) Show that there exists some measurable set $E \in \mathcal{M}$ such that $f = \chi_E$ μ-a.e.

3 [10 points] Let $(X, \mathcal{B})$ be a measurable space and $\langle \mu_n \rangle$ a sequence of finite measures on $(X, \mathcal{B})$ that converge setwise to a finite measure μ (i.e. for each $E \in \mathcal{B}$, $\mu_n(E)$ converges to $\mu(E)$) and $\langle f_n \rangle$ a sequence of nonnegative measurable functions on X that converge pointwise to the function f.

Prove that
$$\int f \, d\mu \leq \liminf \int f_n \, d\mu_n.$$

Hint: Observe that if $\phi \leq f$ is a simple function, then, for all $\epsilon > 0$,
$$X = \bigcup_{n \in \mathbb{N}} \{x : f_k(x) \geq (1 - \epsilon)\phi(x) \text{ for all } k \geq n\}$$

4 [10 points] Let $1 \leq p < \infty$, and let $L^p(\mathbb{R}^n)$ be endowed with its standard norm $\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p}$. Let g be a measurable function on $\mathbb{R}^n$. Define X_g to be the subspace of all $f \in L^p(\mathbb{R}^n)$ such that $fg \in L^p(\mathbb{R}^n)$, and endow X_g with the same norm as L^p. Let $T : X_g \to L^p(\mathbb{R}^n)$ be given by $Tf = fg$.

(a) Show that T is closed.

(b) Show that T is bounded if and only if $g \in L^\infty(\mathbb{R}^n)$.

5 [10 points] Let $f \in L^2(\mathbb{R})$. For each $k \in \mathbb{Z}$ define

$$f_k(x) = \int_k^{k+1} e^{2\pi i x\xi} \hat{f}(\xi)\, \mathrm{d}\xi.$$

Prove that

$$f = \sum_{k\in\mathbb{Z}} f_k,$$

where the sum converges in $L^2(\mathbb{R})$. (Hints: Prove that $\hat{f}\chi_{[k,k+1)} \in L^1$ for each k and apply the inverse Fourier transform. Use Exercise 60 from Chapter 5 in Folland.)

6 [10 points] Let H be a Hilbert space and let $\{x_n\}$ be a sequence that converges *weakly* to x in H.

(a) Show that $\|x\| \leq \liminf \|x_n\|$. (Hint: $\langle x, x \rangle = \lim_{k\to\infty} \langle x_{n_k}, x \rangle$ for any subsequence $\{x_{n_k}\}$.)

(b) If $\|x_n\| \to \|x\|$, show that $x_n \to x$ strongly.

7 [10 points] Let (X, μ) be a finite measure space. Let $\{f_n\}$ be a sequence in $L^p(X, \mu)$ and let $f \in L^p(X, \mu)$. Assume $f_n \to f$ a.e. and that $\|f_n\|_p$ is bounded. Prove that $\|f_n - f\|_q \to 0$ for all $1 \leq q < p$. (Hint: use Egorov's Theorem and Hölder's inequality.)