BAYLOR UNIVERSITY
Department of Mathematics

Ph.D Comprehensive Examination: Applied Math Part II

Summer, 2021

INSTRUCTIONS:

- Do **ALL** problems. This part of the exam takes up to **120** minutes to complete.
- Write your results clearly so that a scanned copy can be emailed.
- You will be graded on how you arrived at the final answer. **Show your detailed work**.

---

1. Consider the numerical solution of the IVP

$$y' = f(t, y), \quad t \geq t_0, \ y(t_0) = y_0, \tag{1.1}$$

where f satisfies a Lipschitz condition in the second variable.

(a) State the definition of well-posedness for (1.1).

(b) Show that the following trapezoidal rule method is convergent for (1.1).

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right], \quad n = 0, 1, 2, \ldots$$

(c) Show that the following implicit Runge-Kutta method for solving (1.1) is A-stable.

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

2. Consider the following IBVP:

$$u_t = u_{xx} + \kappa u_x + f(x), \quad -1 < x < 1, \ t > t_0; \tag{3.1}$$

$$u(-1, t) = u(1, t) = 0, \quad t \geq t_0; \tag{3.2}$$

$$u(x, 0) = \phi(x), \quad -1 < x < 1, \tag{3.3}$$

where $\kappa \in \mathbb{R}$ is a constant and ϕ is sufficiently smooth on $(-1, 1)$.

(a) Derive a semi-discretized scheme for solving (3.1)-(3.3) based on a central finite difference approximation to the spatial derive on uniform grids. Show that your scheme is consistent.

(b) Use the Milne device to derive a posteriori error estimator for your semi-discretized scheme.

(c) Derive a fully discretized scheme method based on your semidiscrezed scheme and a forward Euler method. Show the consistency.

3. Let $y = f(x),\ x \in \mathbb{R}$, be sufficiently smooth and

$$\Delta_+ f(x) = \frac{f(x+p) - f(x)}{p}, \quad \Delta_- f(x) = \frac{f(x) - f(x-q)}{q},$$

where $0 < p, q \ll 1,\ p \neq q$.

   (a) Show that in general $\Delta_+\Delta_- f(x) \neq \Delta_-\Delta_+ f(x)$.

   (b) Show that neither $\Delta_+\Delta_- f(x)$ nor $\Delta_-\Delta_+ f(x)$ is a consistent approximation of the derivative $f''(x),\ x \in \mathbb{R}$.

   (c) **[optional]** Derive a consistent finite difference formula approximating $f''(x)$ utilizing the above steps $p,\ q$ for $x \in \mathbb{R}$.