Algebra Qualifying Exam **Summer 2021**

Instructions: Complete eight of the following problems. Start a new page for each problem.

1. Let G be a group.
   (a) A normal subgroup of G is a subgroup N of G such that ...
   (b) If N is a normal subgroup of G, how are the quotient group G/N and the quotient homomorphism $\pi : G \to G/N$ defined?
   (c) If $\varphi : G \to H$ is a group homomorphism, then $\ker \varphi = \ldots$
   (d) Suppose N is a normal subgroup of G and $\pi : G \to G/N$ is the quotient homomorphism. If $\varphi : G \to H$ is a group homomorphism such that $N \leq \ker \varphi$, show that there exists a unique homomorphism $\bar{\varphi} : G/N \to H$ such that $\bar{\varphi}(aN) = \varphi(a)$.

2. Let G be a group acting on the set X and let $x \in X$.
   (a) The orbit of x, denoted $\mathcal{O}(x)$, is the set ...
   (b) The stabilizer of x, denoted G_x, is the subgroup ...
   (c) State and prove the Orbit-Stabilizer Theorem.
   (d) If $y \in \mathcal{O}(x)$, show that G_x and G_y are conjugate subgroups of G.

3. Let G be a finite group whose order is divisible by the prime p.
   (a) What is a Sylow p-subgroup of G?
   (b) State the three Sylow Theorems.
   (c) If G is non-abelian and simple, explain why G must have more than one Sylow p-subgroup.
   (d) If p and q are distinct primes, show that there is no simple group of order p^2q.

4. Let R be an integral domain
   (a) Define: An element $p \in R$ is irreducible if ...
   (b) Define: An element $p \in R$ is prime if ...
   (c) Show that every prime element $p \in R$ is irreducible.
   (d) If R is a UFD, show that every irreducible element $p \in R$ is prime.

5. Let p be a prime.
   (a) A field F has characteristic p if ...
   (b) Give an example of an infinite field of characteristic p.
   (c) Show that for every positive integer n there exists a finite field of order p^n
   (d) Explain why the field in part (c) is unique (up to isomorphism).

6. Let E be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$.

   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

   (b) Show that $[E : \mathbb{Q}] = 6$.

   (c) Find the Galois group of $E/\mathbb{Q}$.

   (d) Give an example of an intermediate field $\mathbb{Q} \leq K \leq E$ such that $K/\mathbb{Q}$ is *not* a normal extension. Justify your answer.

7. Let R be a commutative ring.

   (a) What does it mean for an R-module M to be finitely generated?

   (b) State the Fundamental Structure Theorem of finitely generated modules over a PID.

   (c) Determine (up to isomorphism) all abelian groups of order 144. Write each group in “invariant factor form” and in “elementary divisor form.”

   (d) Give an example of a finitely generated abelian group that is neither free nor finite.

8. Let R be a commutative ring.

   (a) What is an exact sequence of R-modules?

   (b) A short exact sequence $0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_1} M_3 \longrightarrow 0$ is split exact if ...

   (c) Consider a commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
0 & \longrightarrow & M_1' & \xrightarrow{\varphi_1'} & M_2' & \xrightarrow{\varphi_2'} & M_3' & \longrightarrow & 0
\end{array}$$

   If α_1 and α_3 are isomorphisms, show that α_2 is also an isomorphism.

9. Let M be an R-module.

   (a) The module M is simple if ...

   (b) The module M is semisimple if one of the following three conditions is satisfied: ...

   (c) Give an example of a $\mathbb{Z}$-module of the form $\mathbb{Z}/n\mathbb{Z}$ that is semisimple but not simple. Justify your answer.

   (d) Give an example of a $\mathbb{Z}$-module of the form $\mathbb{Z}/n\mathbb{Z}$ that is not semisimple. Justify your answer.