Instructions: Complete eight of the following problems. Start a new page for each problem.

1. Let $G$ be a group.
   
   (a) A normal subgroup of $G$ is a subgroup $N$ of $G$ such that ...
   
   (b) If $N$ is a normal subgroup of $G$, how are the quotient group $G/N$ and the quotient homomorphism $\pi : G \to G/N$ defined?
   
   (c) If $\varphi : G \to H$ is a group homomorphism, then $\ker \varphi = \ldots$
   
   (d) Suppose $N$ is a normal subgroup of $G$ and $\pi : G \to G/N$ is the quotient homomorphism. If $\varphi : G \to H$ is a group homomorphism such that $N \subseteq \ker \varphi$, show that there exists a unique homomorphism $\overline{\varphi} : G/N \to H$ such that $\overline{\varphi}(aN) = \varphi(a)$.

2. Let $G$ be a group acting on the set $X$ and let $x \in X$.
   
   (a) The orbit of $x$, denoted $O(x)$, is the set \ldots
   
   (b) The stabilizer of $x$, denoted $G_x$, is the subgroup \ldots
   
   (c) State and prove the Orbit-Stabilizer Theorem.
   
   (d) If $y \in O(x)$, show that $G_x$ and $G_y$ are conjugate subgroups of $G$.

3. Let $G$ be a finite group whose order is divisible by the prime $p$.
   
   (a) What is a Sylow $p$-subgroup of $G$?
   
   (b) State the three Sylow Theorems.
   
   (c) If $G$ is non-abelian and simple, explain why $G$ must have more than one Sylow $p$-subgroup.
   
   (d) If $p$ and $q$ are distinct primes, show that there is no simple group of order $p^2q$.

4. Let $R$ be an integral domain
   
   (a) Define: An element $p \in R$ is irreducible if \ldots
   
   (b) Define: An element $p \in R$ is prime if \ldots
   
   (c) Show that every prime element $p \in R$ is irreducible.
   
   (d) If $R$ is a UFD, show that every irreducible element $p \in R$ is prime.

5. Let $p$ be a prime.
   
   (a) A field $F$ has characteristic $p$ if \ldots
   
   (b) Give an example of an infinite field of characteristic $p$.
   
   (c) Show that for every positive integer $n$ there exists a finite field of order $p^n$
   
   (d) Explain why the field in part (c) is unique (up to isomorphism).
6. Let $E$ be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$.
   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
   (b) Show that $[E : \mathbb{Q}] = 6$.
   (c) Find the Galois group of $E/\mathbb{Q}$.
   (d) Give an example of an intermediate field $\mathbb{Q} \leq K \leq E$ such that $K/\mathbb{Q}$ is not a normal extension. Justify your answer.

7. Let $R$ be a commutative ring.
   (a) What does it mean for an $R$-module $M$ to be finitely generated?
   (b) State the Fundamental Structure Theorem of finitely generated modules over a PID.
   (c) Determine (up to isomorphism) all abelian groups of order 144. Write each group in “invariant factor form” and in “elementary divisor form.”
   (d) Give an example of a finitely generated abelian group that is neither free nor finite.

8. Let $R$ be a commutative ring.
   (a) What is an exact sequence of $R$-modules?
   (b) A short exact sequence $0 \rightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \rightarrow 0$ is split exact if ...
   (c) Consider a commutative diagram of $R$-module homomorphisms with exact rows:

   $\begin{array}{ccc}
   0 & \rightarrow & M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \rightarrow 0 \\
   \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
   0 & \rightarrow & M'_1 \xrightarrow{\varphi'_1} M'_2 \xrightarrow{\varphi'_2} M'_3 \rightarrow 0
   \end{array}$

   If $\alpha_1$ and $\alpha_3$ are isomorphisms, show that $\alpha_2$ is also an isomorphism.

9. Let $M$ be an $R$-module.
   (a) The module $M$ is simple if ...
   (b) The module $M$ is semisimple if one of the following three conditions is satisfied: ...
   (c) Give an example of a $\mathbb{Z}$-module of the form $\mathbb{Z}/n\mathbb{Z}$ that is semisimple but not simple. Justify your answer.
   (d) Give an example of a $\mathbb{Z}$-module of the form $\mathbb{Z}/n\mathbb{Z}$ that is not semisimple. Justify your answer.