Topology Qualifier
2019

Name: ____________________________

1. Write *careful* definitions for the following.

   (a) Topology

   (b) Basis

   (c) Closed Set

   (d) Net convergence

   (e) Cluster point for a net

   (f) Continuous function

(g) Compact set

(h) Connected set

(i) Quotient space

(j) Closure of a set

(k) Path connected

(l) A *chain complex* is:

(m) A *short exact sequence* is:

(n) A *simplicial complex* is:

(o) Let X be a topological space. For $p \geq 0$, the *p-th singular chain group* $\mathcal{S}_p(X)$ and *boundary* $\partial : \mathcal{S}_p(X) \to \mathcal{S}_{p-1}(X)$ are given by:

(p) Let Λ be an index set, and for each $\lambda \in \Lambda$, X_λ a topological space. Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. Then the topology on X is *coherent with the topologies on* X_λ if:

(q) A set X in $\mathbb{R}^n$ is *star convex with respect to a point* w if:

(r) A topological space X is a *CW complex* if:

(s) Let X be a CW complex. For $p \geq 0$, the *p-th cellular chain group* $\mathcal{S}_p(X)$ and *boundary* $\partial : \mathcal{S}_p(X) \to \mathcal{S}_{p-1}(X)$ are given by:

(t) The *projective n-space* P^n is defined to be:

2. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) Let X and Y be a topological spaces. A function $f : X \to Y$ is continuous if, and only if, for every net $(x_\lambda)_{\lambda\in\Lambda}$ in X that converges to a point $x \in X$ we have the net $(f(x_\lambda))_{\lambda\in\Lambda}$ converges to $f(x) \in Y$.

   (b) Tychonoff's Theorem

   (c) Let $E \subset X$. If E is connected and $E \subseteq A \subseteq \overline{E}$ then A is also connected.

3. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) Let X be compact. If $\mathcal{E}$ is a collection of closed sets with the FIP, then $\cap\mathcal{E}$ is non-empty.

   (b) Let X be a topological space. If each pair of points $x, y \in X$ is in a set $E_{x,y}$ that is connected, then X is connected.

4. Prove **exactly ONE** of the following theorems from class. You do not need to recopy the statement of the theorem.

   (a) *(Zig-Zag Lemma)* Let $0 \to \mathcal{C} \xrightarrow{\phi_*} \mathcal{D} \xrightarrow{\psi_*} \mathcal{E} \to 0$ be a short exact sequence of chain complexes. Prove that the long sequence of homology groups

   $$\cdots \to H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \to \cdots$$

   is exact. [You may assume that ∂_* is a well-defined homomorphism]

   (b) *The generalized Jordan Curve Theorem* Let $n > 0$ Let C be a subset of $\mathbb{S}^n$ homeomorphic to then $n-1$ sphere. Then $\mathbb{S}^n - C$ has precisely two components, of which C is the common topological boundary.

   (c) *Zero-dimensional Homology* Ket K be a simplical complex. Then the group $H_0(K)$ is free abelian. If $\{v_\alpha\}$ is a collection consisting of a single vertex from each component of $|K|$, then the homology classes of the chains v_α form a basis for $H_0(K)$.

5. Complete **TWO** of the following problems. You must, of course, provide proofs (or counter examples) for your assertions.

   (a) Prove the Brouwer fixed-point theorem, i.e. prove that for $n \geq 0$ every continuous map from B^n to itself has a fixed point.

   (b) Let X be a subspace of $\mathbb{R}^n$ which is star convex relative to the point w. Then X is acyclic in singular homology.

   (c) State the Eilenberg-Steenrod Axioms for homology.

   (d) Let K, L be simplicial complexes and $f, g : K \to L$ simplicial maps that are contiguous. Then there is a chain homotopy between $f_\sharp$ and $g_\sharp$, and hence $f_* = g_*$.

   (e) Prove that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.

6. Calculate the following

   (a) The homology groups of the Klein bottle, K, and the connected sum of two Klein bottles, $K\sharp K$, in all dimensions.

   (b) The fundamental group of the sphere, S^2.