REAL ANALYSIS

Qualifying Exam 2019

Due: 6/14/2019

Instructions:

- This exam is closed book, closed notes.
- Write your solutions clearly and legibly in the space provided; no credit will be given for illegible solutions.
- Turn in scratch paper only if it contains part of your final answer.
- Show all work in your proofs; you will be graded on your reasoning.

Last Name:	**First Name:**

Question	Points	Score
1	6	
2	10	
3	5	
4	5	
5	10	
6	10	
7	10	
8	10	
Total	66	

1 Complete the following statements of definitions and theorems.

(a) [1 point] If X is a topological space, then the *Borel σ-algebra* on X is...

(b) [1 point] Complete the statement of Fatou's Lemma: if $\{f_n\}$ is a sequence of non-negative measurable functions, then...

(c) [1 point] State the Monotone Convergence Theorem.

(d) [1 point] State the Dominated Convergence Theorem.

(e) [1 point] Complete the statement of the Lebesgue-Radon-Nikodym Theorem: if ν is a complex measure and μ is a σ-finite positive measure on $(X, \mathcal{M})$, then there exist...

(f) [1 point] Complete the statement of the Hahn-Banach Theorem: if X is a real vector space, p is a sublinear functional on X, M is a subspace of X, and f is a linear function on M such that $f(x) \leqslant p(x)$ for all $x \in M$, then...

2 [10 points] Solve **one** of the following.

(a) Let m be Lebesgue measure on $\mathbb{R}$. If E is Lebesgue measurable and $0 < m(E) < \infty$, prove that for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$. (Hint: argue by contradiction.)

(b) If μ is a semifinite measure (every set of infinite measure contains a set of positive finite measure) and if $\mu(E) = \infty$, prove that for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$. (Hint: let $M = \sup\{\mu(F) : F \subset E, \mu(F) < \infty\}$. If $M < \infty$, first construct a set $F \subseteq E$ such that $\mu(F) = M$, then use this to derive a contradiction.)

3 [5 points] If $\{f_n\}$ is a sequence of non-negative measurable functions such that f_n decreases pointwise to f and $\int f_1 < \infty$, prove that $\int f = \lim \int f_n$.

4 [5 points] Let μ be a positive measure. A sequence of functions $\{f_n\} \subset L^1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$, then $\left|\int_E f_n \,\mathrm{d}\mu\right| < \epsilon$ for all n. Prove that if $\{f_n\}$ converges to some f in $L^1(\mu)$, then it is uniformly integrable.

5 [10 points] **(a)** State the Closed Graph Theorem.

(b) Let $Y = C([0,1])$, the space of all continuous complex-valued functions on $[0,1]$, and let $X = C^1([0,1])$, the space of all continuously differentiable complex-valued functions on $[0,1]$, both equipped with the uniform norm $\|f\|_u = \sup_{x\in[0,1]}|f(x)|$.

Prove that the derivative map $(d/dx) : X \to Y$ is closed but not bounded. (To show it is closed, use facts from advanced calculus. To show it is not bounded, construct a sequence that is uniformly bounded such that the derivatives blow up.)

(c) Explain why part (b) does not contradict part (a).

6 [10 points] Prove that every closed convex set K in a Hilbert space has a unique element of minimal norm.

7 [10 points] Let $f \in L^2([0,1])$. Prove that $\lim_{n \in \mathbb{N}, n \to \infty} \int_0^1 f(x) \cos(2\pi n x)\, \mathrm{d}x = 0$.

8 [10 points] For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, \mathrm{d}x.$$

Also, let $\Delta = \partial_1^2 + \cdots + \partial_n^2$ denote the usual Laplacian.

Recall that $e^{-\pi|x|^2}$ is its own Fourier transform.

(a) Define

$$G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \; \forall t > 0, \; x \in \mathbb{R}^n.$$

Then for all $t > 0$, show (i) $\hat{G}_t$ is given by $\hat{G}_t(\xi) = e^{-4\pi^2|\xi|^2 t}$, (ii) $\int G_t(x) \, \mathrm{d}x = 1$, and (iii) $\partial_t \hat{G}_t(\xi) = \widehat{\Delta G_t}(\xi)$.

(b) Let $f \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$. Using part (a), sketch the proof showing that $u(x,t) := f * G_t(x)$ (where $*$ denotes convolution) satisfies the initial value problem

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & \text{if } x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x) & \text{if } x \in \mathbb{R}^n \end{cases}$$

where the initial condition holds in the sense that $\left\| u(\cdot, t) - f \right\|_p \to 0$ as $t \to 0$.