BAYLOR UNIVERSITY

Department of Mathematics

Ph.D Comprehensive Examination: Applied Math Part II

Summer, 2019

<u>Instructions</u>:

- Do **3** of the following 6 problems. This part of the exam takes up to 120 minutes to complete.
- Write your results clearly so that a scanned copy can be emailed.
- You will be graded on how you arrived at the final answer. **Show your detailed work**.

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1. Consider the numerical solution of the IVP

$$y' = f(t, y), \quad t \geq t_0,\ y(t_0) = y_0, \tag{1.1}$$

wher f is sufficiently smooth. Denote $t_{n+1} = t_n + h,\ n = 0, 1, 2, \ldots,\ 0 < h \ll 1$, and assume that $y_m \approx y(t_m)$ for $m = 1, 2, \ldots$ Let

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right], \quad n = 0, 1, 2, \ldots \tag{1.2}$$

(a) Show that (1.2) is a consistent approximation of (1.1).

(b) Show that the numerical method (1.2) is convergent. Derive its order of convergence.

2. Given the following order p and convergent multistep method for solving (1.1):

$$\sum_{m=0}^{s} \alpha_m y_{n+m} = h \sum_{m=0}^{s} \beta_m f(t_{n+m}, y_{n+m}), \quad n = 0, 1, 2, \ldots,\ a_s = 1,\ s \geq 1. \tag{2.1}$$

Derive a proper Milne device for assessing its local error.

3. Let $T = (t_{k-j})_{k,j=1}^{n}$ be an $n \times n$ TST matrix with $t_{-n+1} = \cdots = t_{-2} = 0,\ t_{-1} = \beta,\ t_0 = \alpha,\ t_1 = \beta,\ t_2 = \cdots = t_{n-1} = 0$.

(a) Show that the eigenvalues of T are

$$\lambda_j = \alpha + 2\beta \cos\left(\frac{\pi j}{n+1}\right), \quad j = 1, 2, \ldots, n.$$

(b) Use the above result to show that there exists a conditionally stable explicit finite difference method for solving the IBVP:

$$\begin{aligned} &u_t = u_{xx}, \quad -1 < x < 1,\ t > t_0; \\ &u(-1, t) = u(1, t) = 0, \quad t \geq t_0; \\ &u(x, 0) = \phi(x), \quad -1 < x < 1, \end{aligned}$$

where ϕ is sufficiently smooth on $(-1, 1)$. If the stability is conditional, find the condition.

4. Consider the two-dimensional Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = f(x, y), \quad (x, y) \in \Omega, \tag{4.1}$$

where $\Omega = \{(x, y) |\ 0 < x, y < 1\}$, together with the homogeneous Dirichlet boundary condition

$$u(x, y)|_{\partial\Omega} = 0. \tag{4.2}$$

Let Ω_h be a uniform mesh region over Ω and its boundary $\partial\Omega$ with a uniform step size $0 < h \ll 1,\ (n + 1)h = 1$, used in both directions. Consider the standard 5-point scheme for solving the problem:

$$u_{k-1,j} + u_{k+1,j} + u_{k,j-1} + u_{k,j+1} - 4u_{u_k,j} = h^2 f_{k,j}, \quad 1 \le k, j \le n. \tag{4.3}$$

(a) Show that (4.3) can be rewritten as a matrix equation

$$Au = h^2 f.$$

(b) Show that, following a standard order arrangement of the components in u, the matrix A has the eigenvalues

$$\lambda_{\alpha,\beta} = -4\left[\sin^2\left(\frac{\alpha\pi}{2(n+1)}\right) + \sin^2\left(\frac{\beta\pi}{2(n+1)}\right)\right]$$

and eigenvectors $v_{\alpha,\beta} = \{v_{\alpha,\beta,i,j}\}$, in which

$$v_{\alpha,\beta,i,j} = \sin\left(\frac{i\alpha\pi}{n+1}\right)\sin\left(\frac{j\beta\pi}{n+1}\right), \quad i, j = 1, 2, \ldots, n,$$

for $\alpha, \beta = 1, 2, \ldots, n$.

5. Consider the solution of the linear system

$$Ax = b, \tag{5.1}$$

where $A \in \mathbb{R}^{n\times n},\ n \gg 1$.

(a) Derive the Jacobi, Gauss-Seidel and SOR iterative schemes, respectively, for solving (5.1).

(b) Under what condition the above-mentioned iterative methods converge?

(c) Show that if A is irreducible and strictly diagonally dominant then the Jacobi method converges.

6. Consider the Cauchy problem

$$w_t = w_{xx}, \ -\infty < x < \infty,\ t > 0; \quad w(x, 0) = \phi(x), \ -\infty < x < \infty. \tag{6.1}$$

Use the von Neumann analysis to discuss the numerical stability of the following finite difference scheme,

$$w_k^{n+1} = w_k^n + \mu\left(w_{k+1}^{n+1} - 2w_k^{n+1} + w_{k-1}^{n+1}\right), \quad n \ge 0; \quad u_k^0 = \phi_k,$$

where $\mu > 0$ is the Courant number and ϕ is sufficiently smooth, for solving (6.1).